VIBRATIONS OF BEAMS ON ELASTIC FOUNDATIONS AND COLUMNS CARRYING CONCENTRATED MASSES

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CERTIFICATE

It is certified that the work contained in the thesis entitled, **"VIBRATIONS OF BEAMS ON ELASTIC FOUNDATIONS AND COLUMNS CARRYING CONCENTRATED MASSES"** been carried out by Mr. V.S. Phani Kanth under my supervision and this work has not been submitted elsewhere for the award of a degree.

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SYNOPSIS

In the present work, solutions for the response of beams carrying concentrated mass, resting on soil medium have been presented idealizing the soil medium as Winkler model. Damping has also been included in these models. The beam with concentrated mass attached to it and resting on elastic foundation has been analysed for steady state dynamic force. The displacement magnitudes have been calculated for beams with different values of concentrated mass attached to it at different positions. It has been found, from the analysis, that as the concentrated mass increases the maximum displacement at the center of the beam decreases, for various positions of the concentrated mass. A general computer program has been presented for this beam. A study is also carried out for columns carrying concentrated mass at the centre of the column. A general computer program has also been given for these columns.

NOMENCLATURE

A	Cross sectional area of the beam
c	Damping constant
c_c	Critical damping constant
D	Damping factor
E	Elastic modulus of the beam
f	Coulomb's frictional force
F_o	Force of constant magnitude
$F_{t+\theta\Delta t}$	Force vector at time $t + \theta\Delta t$
I	Moment of inertia of the beam
K	Spring constant of the soil medium
l	Length of the finite element
L	Total length of the beam/column
m	Concentrated mass attached to the beam/column
[N]	Interpolation matrix
$[N^{I}]$	First derivative of N with respect to x
$[N^{II}]$	Second derivative of N with respect to x
q(x)	Foundation reaction in vertical direction at a distance x
q(x,t)	Foundation reaction in vertical direction at a distance x at time t
u	Axial displacement of the column
x	Distance parameter
X	Coordinate along the length of the beam/column

y	Deflection in vertical direction
Δt	critical time step
α	Concentrated mass ratio
ρA_c	Mass density of the soil media
$\lfloor \; \rfloor$	Row matrix
$\{ \; \}$	Column matrix

CHAPTER 1

INTRODUCTION

1.1 General

The dynamic analysis of foundations finds many applications in modern technology such as power plants, machine tools,railway tracks, aircraft runways, buried pipes, rocket testing tracks etc. Such problems of foundation-soil interaction are generally analyzed by incorporating the reaction from the foundation into the response mechanism of the structure by idealizing the foundation by a suitable mathematical model. In majority of the cases, the response of the structure at the contact surface is of prime interest. Thus it would be of immense help in the analysis, if the foundation can be represented by a simple mathematical model with the Parameters of the model characterizing the true behavior of the system as closely as possible. To accomplish this objective, many foundation models, from the Winkler foundation to the elastic continuum idealization have been evolved. A comprehensive review of some of these foundation models is given by Kerr(1964).

The dynamic loading effects on stationary structure have been the subject of great interest. A lot of work has been done on the behavior of beams subjected to steady state forces and transient loads. But for beams, resting on soil media the available literature is quite meagre. In technical applications it is not infrequent that a beam resting on soil media which is under

vibration carries a lumped or concentrated mass somewhere along the length of the beam. A machine resting on a deep beam is an example. No solutions have been reported so far on such problems also, when the machine is eccentric to the beam. Thus it will be very useful in the analysis of the problems to consider the effect of the concentrated mass.

1.2 Brief Review of Literature

The earliest formulation of elastic foundation was due to Winkler who assumed that the foundation model consisted of closely spaced independent linear springs Kerr(1964). The Winkler model approximates the reaction of the continuum by a simple expression which results in a single ordinary differential equation governing the responses of the beam. Several other authors have evolved various Partly continuous models conceptually developed from the Winkler model, such as Pasternak, Filonenko Borodich, Weighardt, Hetenyi Kerr etc., [Kerr(1964)]. The objectives in employing a model are mathematical simplicity and relatively accurate representation of the real system. Often, the visco-elastic model (Spring-Dashpot System) is used in dynamic analysis owing to mathematical simplicity, though the above mentioned models which are used for static loads, can also be used for dynamic analysis with appropriate inclusion of damping. In visco-elastic model, the soil medium is idealized in terms of physical constants- such as spring constant, damping coefficient and the equivalent mass of Soil participating in the vibration. The damping is attributed to radiation of energy due to the Propagation of the waves in the

medium, and the energy dissipation through inter-granular friction. Through experimental evidence, Kausel and Rosset(1975) indicated that the energy dissipation takes place largely due to internal friction. It was also indicated that this energy is independent of frequency. about the spring constant, most of the methods presume that the spring constant is independent of motion. This assumption is not strictly valid [Whitman(1964)]. However, so long as the motions are within certain limits, the behaviour of foundations may be taken as linear.

Theoretical investigations on beam vibrations find its origin in Curvature-bending relationship derived by Bernoulli.

For the transient loads, Stadler and Shreeves(1970), Rades (1970) presented the analytical solutions for infinite and finite beam respectively for the dynamic response of the Euler-Bernoulli beam including the effect of damping, axial load or shear layer etc., as given by equation

$$EI\frac{\partial^4 y}{\partial x^4} \pm T\frac{\partial^2 y}{\partial x^2} + Ky + c\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = q(x,t) \qquad (1.1)$$

where T is axial load or shear layer coefficient.

For beams an elastic foundation, some analytical results are available where the foundation modulii are non-linear [Funstone and Hall(1967)].

The problem of lateral vibration of a simply supported beam with concentrated mass attached at the centre was studied by Hoppman II(1952), taking the Euler-Bernoulli beam. Chen(1963) presented a new formulation in which the dirac-delta function is

used in differential equation. Frequency equations for normal modes of vibration were obtained by Baker(1964). Srinath and Das(1967)presented the analysis for simply supported beams having rotary inertia. Later Das and Deshmukh(1970) derived a general expression for Timoshenko beam with concentrated mass at the Centre. Jain(1977) derived a general expression for the response of beams on elastic foundations, with concentrated mass attached to it.

1.3 Scope of the Present Investigation

The present work is on the study of beams resting on soil media. The soil media is idealized as Winkler model with damping included in the model, for the analysis of Euler-Bernoulli beams. A beam with concentrated mass attached to it is analysed for transient vibrations, using finite element analysis. These solutions which can be used for any boundary conditions arechecked with the existing solutions for beams carrying a concentrated mass with end supports. Variation of magnitude of displacement along the length of the beam for various values of concentrated mass, mass position, is presented in this work. Comparisons have been made with some existing solutions.

Also, the effect of concentrated mass on the axially loaded element is investigated.

The general conclusions and scope for further work have been presented in chapter 4.

Table 1.1

Criteria for flexibility of beams

S.No.	Size of beam	Criteria for distinction	Procedure recommended
1.	Long beam	$\lambda L > 0.5$	Winkler analysis assuming infinite beam
2.	Moderately long	$2.25 < \lambda L < 5.0$	Winkler analysis
3.	Short moderate beam	$0.8 < \lambda L < 2.25$	De beer method
4.	Short beam	$\lambda L < 0.8$	Treat as perfectly rigid.

CHAPTER - 2

DYNAMIC RESPONSE OF BEAMS ON ELASTIC FOUNDATION CARRYING A CONCENTRATED MASS

2.1 Introduction

This chapter deals with the finite element solution of Euler-Bernoulli beam, carrying a concentrated mass attached to it, resting on elastic foundation. A machine resting on deep beam, a column resting on strip footing, a mechanical oscillator (to produce a high force oscillation) on footing, for the determination of soil properties are the few examples which simulate this problem. Also this chapter deals with, an axially loaded column on elastic foundation.

In the literature, solutions for beams carrying concentrated mass and simply supported at the ends without elastic foundation are available [Hopmann(1952), Chen(1963), Baker(1964), Srinath and Das(1967), Das and Deshmukh(1970), Kameswara Rao and Das(1975)] and for beams carrying concentrated mass with elastic foundation has been given in [Jain(1977)]. However the solution by Jain(1977), does not include the effects of eccentric concentrated mass. In this chapter a finite element formulation has been given for beams resting an elastic foundation, with Winkler model idealisation with damping included. Similarly, a finite element formulation for columns and carrying a concentrated mass has been

given.

2.2 General Theory

The differential equation for transverse vibration of an Euler-Bernoulli beam on elastic foundation subjected to a dynamic load can be written as

$$EI \frac{\partial^4 y}{\partial x^4} + ky + c \frac{\partial y}{\partial t} + \rho A_c \frac{\partial^2 y}{\partial t^2} = q(x,t) \tag{2.1}$$

where

E is Young's Modulus of Elasticity,

I is Moment of Inertia of the beam,

k is Spring Constant of the soil medium,

c is damping coefficient of the soil medium,

ρA_c is mass density of the soil medium,

y is vertical displacement of the beam,

EI is Modulus of rigidity of the beam,

and q(x,t) is the dynamic forcing function.

The above fourth order linear Partial differential equation is solved using the the finite element technique. As stated earlier, the soil is idealized by Winkler assumption with the effect of damping included. In the Winkler model it is assumed that the foundation applies only a reaction p(x) normal to the beam and that p(x) is directly proportional to the deflection i.e., p(x) = k y(x), where k is the Winkler foundation modulus in

2.3 Finite element analysis

For the finite element solution of Eq.2.1, the beam is divided into n number of finite elements as shown in Fig.2.1. A typical finite element is shown in Fig.2.2.

Now, converting the Eq.2.1 into variational form,

$$\int_0^l \left[EI\frac{\partial^4 y}{\partial x^4} + ky + c\frac{\partial y}{\partial t} + \rho A_c \frac{\partial^2 y}{\partial t^2} - q \right] \delta y \, dx = 0 \tag{2.2}$$

$$\int_0^l EI\,\frac{\partial^4 y}{\partial x^4}\,\delta y\,dx = EI\,\frac{\partial^3 y}{\partial x^3}\,\delta y\Bigg|_0^l - \int_0^l EI\,\frac{\partial^3 y}{\partial x^3}\,\delta\left(\frac{\partial y}{\partial x}\right)dx$$

$$= EI\,\frac{\partial^3 y}{\partial x^3}\,\delta y\Bigg|_0^l - EI\,\frac{\partial^2 y}{\partial x^2}\,\delta\left(\frac{\partial y}{\partial x}\right)\Bigg|_0^l + \int_0^l EI\,\frac{\partial^2 y}{\partial x^2}\,\delta\left(\frac{\partial^2 y}{\partial x^2}\right)dx$$

$$= \delta \int_0^l \frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx + EI\,\frac{\partial^3 y}{\partial x^3}\,\delta y\Bigg|_0^l - EI\,\frac{\partial^2 y}{\partial x^2}\,\delta\left(\frac{\partial y}{\partial x}\right)\Bigg|_0^l$$

...(2.3)

Similarly,

$$\int_0^l Ky\,\delta y\,dx = \delta \int_0^l \frac{k}{2}\,y^2\,dx$$

$$\int_0^l \rho A_c\,\frac{\partial^2 y}{\partial t^2}\,\delta y\,dx = \delta \int_0^l \rho A_c\,\frac{\partial^2 y}{\partial t^2}\,y\,dx$$

$$\int_0^l c\,\frac{\partial y}{\partial t}\,\delta y\,dx = \delta\int_0^l c\,\frac{\partial y}{\partial t}\,y\,dx \tag{2.4}$$

therefore the Eq.2.1 becomes

$$\delta\left[\int_0^1 \left(\frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2}\right)^2 + 1/2\ K\ y^2 + \rho A_c \frac{\partial^2 y}{\partial t^2}\ y + C\frac{\partial y}{\partial t}\ y - qy\right) dx\right.$$

$$\left. + \quad EI\ \frac{\partial^3 y}{\partial x^3}\ \delta y\Big|_0^1 \ - \ EI\ \frac{\partial^2 y}{\partial x^2}\ \delta\left(\frac{\partial y}{\partial x}\right)\Bigg|_0^1\right] = 0 \qquad (2.5)$$

i.e.,

$$\delta I = 0$$

where

$$I = \left[\int_0^1 \left(\frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2}\right)^2 + 1/2\ K\ y^2 + \rho A_c \frac{\partial^2 y}{\partial t^2}\ y + C\frac{\partial y}{\partial t}\ y - qy\right) dx\right.$$

$$\left. + \quad EI\ \frac{\partial^3 y}{\partial x^3}\ y\Big|_0^1 \ - \ EI\ \frac{\partial^2 y}{\partial x^2}\left(\frac{\partial y}{\partial x}\right)\Bigg|_0^1\right] \qquad (2.6)$$

Now its solution is assumed as,

$$y^e = \lfloor N_1 \quad N_2 \ldots\ldots N_i \ldots\ldots N_r \rfloor \begin{Bmatrix} y_1 \\ . \\ \dot{y}_i \\ \dot{y}_r \end{Bmatrix}$$

$$= \lfloor N \rfloor \{y\}^{ne} \qquad (2.7)$$

where, N_i is the interpolating function of the element.

Y_i is nodal parameter of the element

r is the degree of freedom of the element

also

$$(y^e)' = \lfloor N' \rfloor\ \{y\}^{ne} \qquad (2.8)$$

$$(y^e)'' = \lfloor N'' \rfloor\ \{y\}^{ne} \qquad (2.9)$$

$$(\dot{y}^{e}) = \lfloor N \rfloor \{\dot{y}\}^{ne} \qquad (2.10)$$

$$(\dot{y}^{\cdot})^{e'} = \lfloor N \rfloor \{\dot{y}^{\cdot}\}^{ne} \qquad (2.11)$$

The superscripts *dashes*, as shown above indicate the derivatives with respect to x, where as the superscripts *dots* represent the derivatives with respect to time.

Now, the highest derivative of y in Eq.2.6 is of order two in the integral. Thus the interpolating function N_i should have a compatibility of $y^{(e)}$ and $\partial y^{(e)}/\partial x$ i.e., the slope and deflection at the element interfaces should be continuous. The highest derivative in Eq.2.6 is of order three. Hence the interpolating function N_i should have a completeness of order three.

$$y^{(e)} = a + bx + cx^2 + dx^3 \qquad (2.12)$$

Upon applying the boundary conditions at element ends ($y = y_j$, $y' = y'_j$ @ $x = 0$ and $y = y$, $y' = y'$ @ $x = 1$), the Eq.2.12 becomes

$$y^{(e)} = \lfloor N_1 \; N_2 \; N_3 \; N_4 \rfloor \begin{Bmatrix} y_j \\ y'_j \\ y_k \\ y'_k \end{Bmatrix} \qquad (2.13)$$

where,

$$N_1 = 1-3\,(x/1)^2 + 2\,(x/1)^3$$

$$N_2 = 1\,((x/1) - 2(x/1)^2 + (x/1)^3)$$

$$N_3 = 3(x/1)^2 - 2\,(x/1)^3$$

and

$$N_4 = 1((x/1)^3 - (x/1)^2) \qquad (2.14)$$

Hence,

$$I = \int_{o}^{1} 1/2\ EI\ \lfloor y \rfloor^{(ne)} \{N''\}\ \lfloor N'' \rfloor\ \{y\}^{(ne)}\ dx$$

$$+ \int_{o}^{1} 1/2\ K\ \lfloor y \rfloor^{(ne)}\ \{N\} \lfloor N \rfloor \{y\}^{(ne)} dx$$

$$+ \int_{o}^{1} \rho A_c\ \lfloor N \rfloor\ \{\dot{y}^{\cdot}\}^{(ne)} \lfloor N \rfloor \{y\}^{(ne)} dx$$

$$+ \int_{o}^{1} c \lfloor N \rfloor \{\dot{y}\}^{(ne)}\ \lfloor N \rfloor\ \{y\}^{(ne)}\ dx$$

$$- \int_{o}^{1} q\ \lfloor N \rfloor\ \{y\}^{(ne)} dx$$

$$- \lfloor y_j \quad y'_j \quad y_k \quad y'_k \rfloor \begin{Bmatrix} EI \dfrac{\partial\ y^3}{\partial x^3} \Big|_{x=0} \\ - EI \dfrac{\partial^2 y}{\partial x^2} \Big|_{x=0} \\ - EI \dfrac{\partial^3 y}{\partial x^3} \Big|_{x=1} \\ + EI \dfrac{\partial^2 y}{\partial x^2} \Big|_{x=1} \end{Bmatrix} \qquad (2.15)$$

We have the condition,

$$\frac{\partial I}{\partial \lfloor y \rfloor^{(ne)}} = 0$$

$$\int_{o}^{1} (EI\ \{N''\}\ \lfloor N'' \rfloor + k\ \{N\}\ \lfloor N \rfloor)\ \{y\}^{(ne)} dx$$

$$+ \int_{o}^{1} \rho A_c\ \{N\}\ \lfloor N \rfloor\ \{\dot{y}^{\cdot}\}^{(ne)}\ dx$$

$$+ \int_0^l c \{N\} \lfloor N \rfloor \, dx \, \{\dot{y}\}^{(ne)}$$

$$- \int_0^l q \{N\} \, dx - \{F_B\} = 0 \qquad (2.16)$$

$$\text{where } F_B = \begin{Bmatrix} EI \left. \dfrac{\partial^3 y}{\partial x^3} \right|_{x=0} \\ -EI \left. \dfrac{\partial^2 y}{\partial x^2} \right|_{x=0} \\ -EI \left. \dfrac{\partial^3 y}{\partial x^3} \right|_{x=l} \\ EI \left. \dfrac{\partial^2 y}{\partial x^2} \right|_{x=l} \end{Bmatrix}$$

$$[M]^{(e)} \{\ddot{y}\}^{(ne)} + [c]^{(e)} \{\dot{y}\}^{(ne)} + [K]^{(e)} \{y\}^{(ne)} = \{F\} \qquad (2.17)$$

$$[M]^{(e)} = \int_0^l \rho A_c \{N\} \lfloor N \rfloor \, dx$$

$$= \frac{\rho A_c l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ & \text{symmetric} & & 4l^2 \end{bmatrix}$$

$$[c]^{(e)} = \int_0^l c \{N\} \lfloor N \rfloor \, dx$$

$$= \frac{cl}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ & \text{symmetric} & & 4l^2 \end{bmatrix}$$

$$\frac{kl}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{symmetric} & & & 4l^2 \end{bmatrix}$$

Now this procedure is carried out for every element and the equation is to be assembled finally to bring it down to the form

$$[M]\{\ddot{y}\} + [c]\{y\} + [K]\{y\} = \{F\} \tag{2.18}$$

There are two ways of solving Eq.2.18 :

1. Direct Integration techniques and
2. Mode superposition technique

Direct Integration Techniques :

In this approach recurrence relations in time domain are used and depending on the type of relation various methods have been evolved as mentioned below.

(a) Forward difference scheme,

(b) Central difference scheme,

(c) Backward difference scheme,

(d) Houbolt method,

(e) Wilson-θ method, and

(f) Newmark's method

The methods (a) to (c) are explicit methods, and the methods (d) to (f) are implicit type. For these methods stability and convergence of the solution depends upon the time step chosen.

In Mode super position, decoupled equations are obtained and then numerical integration is carried out. The Wilson-θ method is

used in the present work. Being an implicit method, this method is unconditionally stable, and hence, there is no critical time step and Δt can, in general, be selected many times larger than for central difference method.

2.4 Wilson-θ Method

This method is essentially an extension of the linear acceleration method in which a linear variation of acceleration from time t_n to time t_{n+1} is assumed. In the Wilson-θ method the acceleration is assumed to be linear from time t to time $t + \theta\Delta t$, where $\theta \geq 1$, as shown in Fig.2.3.

For $\theta = 1.0$ the method reduces to the linear acceleration scheme but in this case the method is only conditionally stable. For unconditional stability $\theta \geq 1.37$. For any time τ so that $0 \leq \tau \leq \theta\ \Delta t$. We have from Fig.2.3 that

$$\ddot{y}_{t+\tau} = \ddot{y}_t + \frac{\tau}{\theta\ \Delta t}\left(\ddot{y}_{t+\theta\ \Delta t} - \ddot{y}_t\right) \tag{2.19}$$

Integrating Eq.(2.19) gives

$$\ddot{y}_{t+\tau} = \dot{y}_t + \ddot{y}_t\ \tau + \frac{\tau^2}{2\ \theta\ \Delta t}\left(\ddot{y}_{t+\theta\Delta t} - \ddot{y}_t\right) \tag{2.20}$$

and Integrating again gives

$$y_{t+\tau} = y_t + \dot{y}_t\ \tau + \frac{1}{2}\ \ddot{y}_t\tau + \frac{\tau^3}{6\ \theta\ \Delta t}\left(\ddot{y}_{t+\theta\Delta t} - \ddot{y}_t\right) \tag{2.21}$$

or for the particular time $\tau = \theta\ \Delta t$, Eqs. 2.19 and 2.20 give

$$\dot{y}_{t+\theta\Delta t} = \dot{y}_t + \frac{\theta\ \Delta t}{2}\left(\ddot{y}_{t+\theta\Delta t} + \ddot{y}_t\right) \tag{2.22}$$

from which we can solve for $\ddot{y}_{t+\theta\Delta t}$ and $\dot{y}_{t+\theta\Delta t}$ in terms of $y_{t+\theta\Delta t}$:

$$\ddot{y}_{t+\theta\Delta t} = \frac{6}{(\theta\ \Delta t)^2}(y_{t+\theta\Delta t} - y_t) - \frac{6}{(\theta\ \Delta t)}\dot{y}_t - 2\ \ddot{y}_t \tag{2.23}$$

$$\dot{y}_{t+\theta\Delta t} = \frac{3}{(\theta\ \Delta t)}(y_{t+\theta\Delta t} - y_t) - 2\ \dot{y}_t - \frac{\theta\ \Delta\ t}{2}\ \ddot{y}_t \tag{2.24}$$

To obtain the displacements, velocities and accelerations at time $t+\Delta t$, the equilibrium equations are considered at time $t+\theta\Delta t$. This requires projection of the applied load vector to time $t + \theta\ \Delta t$, which is performed linearly as

$$\mathbf{F}_{t+\theta\Delta t} = \mathbf{F}_t + \theta\ (\ \mathbf{F}_{t+\Delta t} - \mathbf{F}_t) \tag{2.25}$$

and then Eq.(2.17) at $t + \theta\Delta t$ becomes

$$[M]\ \{\ \ddot{y}\ \}_{t+\theta\Delta t} + [C]\{\dot{y}\}_{t+\theta\Delta t} + [K]\{y\}_{t+\theta\Delta t} = \{\mathbf{F}\}_{t+\theta\Delta t} \tag{2.26}$$

Substitution of Eq.(2.23) and (2.24) into Eq.(2.26) gives the following system of simultaneous equations, which may be solved to give $y_{t+\theta\Delta t}$ as :

$$\begin{aligned}\left\{b_o[M] + b_1\ [C] + [K]\right\}\{y\}_{t\ +\ \theta\Delta t} &= \{F\}_t + \theta\ (\ F_{t\ +\ \theta\Delta t} - F_t) \\ &+ [M]\ (b_o\ y_t + b_2\ \dot{y}_t + b_3\ \ddot{y}_t) \\ &+ [C]\,(b_1 y_t + b_4 \dot{y}_t + b_5\ \ddot{y}_t)\end{aligned} \tag{2.27}$$

where

$b_o = 6/\tau^2$; $b_1 = 3/\tau$; $b_2 = 2b_1$; $b_3 = 2$; $b_4 = 2$; $b_5 = \tau/2$

Substituting $y_{t+\theta\Delta t}$ into Eq.(2.23) gives $\ddot{y}_{t+\theta\Delta t}$, which is then employed in Eqs.2.18 through 2.20, all evaluated at $\tau = \Delta t$. to give

$$\ddot{y}_{t+\Delta t} = b_6(y_{t+\theta\Delta t} - y_t) + b_7(\dot{y})_t + b_8\ddot{y}_t$$

$$\dot{y}_{t+\Delta t} = \dot{y}_t + b_9(\ddot{y}_{t+\Delta t} + \ddot{y}_t)$$

$$y_{t+\Delta t} = y_t + \Delta t\ \dot{y}_t + b_{10}(\ddot{y}_{t+\Delta t} + 2\ddot{y}_t) \qquad (2.28)$$

where

$$b_6 = b_o / \theta\ ;\ b_7 = -b_2 / \theta\ ;\ b_8 = 1 - 3/\theta$$

$$b_9 = \Delta t/2\ ;\ b_{10} = \Delta t^2 / 10$$

2.5 Effect of Concentrated Mass

Now, for a beam carrying a concentrated mass and subjected to any generalized kind of transverse loading a finite element programme is developed. Fig.2.4 shows a beam carrying a concentrated mass and resting on an elastic foundation, subjected to a steady state constant exciting force $F_o\ e^{iwt}$. The inertia force is $m\ddot{y}$. If this mass acts at element i, on nodes j, k, the elemental matrix due to this inertial force is $\int_0^1 m\ \ddot{y}\{N\}\ dx$. For every element, the same procedure is carried out and is assembled finally. This assembled matrix is taken into mass matrix [m] on the left hand side, and the terms are added correspondingly.

2.6 Finite Element Analysis of the Column

From the Fig.2.5, the equilibrium of the element in the horizontal direction is,

$$EA\frac{\partial^2 u}{\partial x^2}\ dx - m\frac{\partial^2 u}{\partial t^2} + q\ dx - f dx = 0$$

i.e,

$$EA\frac{\partial^2 u}{\partial x^2} - m\ \frac{\partial^2 u}{\partial x^2} + q - f = 0 \qquad (2.30)$$

A(x) is the area per unit length

m(x) is mass per unit length

$u(x,t)$ is displacement at any point x

f is the coulomb friction due to soil per unit length

Converting Eq.(2.30) into variational form

$$\int_0^1 \left(EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} + q - f \right) \delta U \, dx = 0 \tag{2.31}$$

$$EA \frac{\partial u}{\partial x} \Big|_0^1 \delta u - \int_0^1 EA \frac{\partial u}{\partial x} \delta \left(\frac{\partial u}{\partial x} \right) dx$$
$$- \int_0^1 \rho A \frac{\partial^2 u}{\partial t^2} \delta u \, dx + \int_0^1 (q-f) \, \delta u \, dx = 0 \tag{2.32}$$

$$EA \frac{\partial u}{\partial x} \delta u \Big|_0^1 - \frac{1}{2} \int_0^1 EA \, \delta \left(\frac{\partial u}{\partial x} \right)^2 dx$$
$$- \int_0^1 \rho A \frac{\partial^2 u}{\partial t^2} \delta u \, dx + \int_0^1 (q-f) \, \delta u \, dx = 0 \tag{2.33}$$

$$\delta \left(- \frac{1}{2} \int_0^1 EA \left(\frac{\partial u}{\partial x} \right)^2 dx - \int_0^1 \rho A \frac{\partial^2 u}{\partial t^2} u \, dx + \int_0^1 (q-f) \, u \, dx \right.$$
$$\left. + EA \frac{\partial u}{\partial x} u \Big|_0^1 \right) = 0 \tag{2.34}$$

i.e $\delta I = 0$

where

$$I = \frac{1}{2} \int_0^1 EA \left(\frac{\partial u}{\partial x} \right)^2 dx + \int_0^1 \rho A \frac{\partial^2 u}{\partial t^2} u \, dx - \int_0^1 (q-f) \, u \, dx$$
$$- EA \frac{\partial u}{\partial x} u \Big|_0^1 = 0 \tag{2.35}$$

Now, its solution is assumed as,

$$\{u\}^e = \lfloor N_1 \; N_2 \; N_3 \ldots N_r \rfloor \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \cdot \\ \cdot \\ \cdot \\ U_r \end{Bmatrix}$$

$$= \lfloor N \rfloor \{U\}^{ne} \tag{2.36}$$

On completeness and compatiability considerations, $u^e = a + bx$, which upon applying boundary conditions at the element ends ($u = u_j$ at $x = 0$ and $u = u_k$ at $x = l$), becomes

$$u^e = \lfloor N_1 \quad N_2 \rfloor \begin{Bmatrix} u_j \\ u_k \end{Bmatrix} \tag{2.37}$$

where,

$$N_1 = 1 - (x/l) \text{ and } N_2 = x/l$$

Hence,

$$I = \frac{1}{2} \int_o^l EA \lfloor u \rfloor^{ne} \{N'\} \lfloor N' \rfloor \{u\}^{ne} dx$$

$$+ \int_o^l \rho A \lfloor N \rfloor \{\ddot{u}\}^{ne} \lfloor N \rfloor \{u\}^{ne} dx + \int_o^l (f-q) \lfloor N \rfloor \{u\}^{ne}$$

$$+ [u_j \quad u_k] \begin{Bmatrix} EA \dfrac{\partial u}{\partial x} \Big|_j \\ -EA \dfrac{\partial u}{\partial x} \Big|_k \end{Bmatrix} \tag{2.38}$$

we have the condition

$$\frac{\partial I}{\partial \lfloor u \rfloor^{ne}} = 0$$

$$\int_o^l EA \{N'\} \lfloor N' \rfloor dx \{u\}^{ne} + \int_o^l \rho A \{N\} \lfloor N \rfloor dx \{\ddot{u}\}^{ne} + \int_o^l (f-q) \{N\} dx = \begin{Bmatrix} -EA \dfrac{\partial u}{\partial x} \Big|_j \\ EA \dfrac{\partial u}{\partial x} \Big|_k \end{Bmatrix} \tag{2.39}$$

$$\frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$-\frac{q}{2}\begin{Bmatrix} 1-\overline{x}/l \\ \overline{x}/l \end{Bmatrix} + \frac{fl}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -EA\left.\frac{\partial u}{\partial x}\right|_j \\ EA\left.\frac{\partial u}{\partial x}\right|_k \end{Bmatrix} \qquad (2.40)$$

Where q is concentrated load and f is uniformly distributed load.This multi degree coupled linear differential equation, can be solved on the same lines as has been discussed in the earlier section to obtain the dynamic response.

CHAPTER III

RESULTS AND DISCUSSION

3.1 Introduction

This chapter deals with the numerical results obtained for beams and columns on elastic foundations carrying concentrated mass. In certain situations the length of the beam may be very long compared to its other dimensions viz., rocket test tracks. When the ends of the beam are far away from the load position, it may be assumed that the effect of load is negligible at the ends. This situation can be idealised as infinite beam.

Response of beams and columns on elastic foundation carrying concentrated mass is considered in the present chapter. The solutions have been obtained by using Finite Element Method (FEM). The load considered is a steady state constant exciting force. The programme developed can also take into account the earthquake kind of loads. Results obtained are compared with solutions by Jain[11] for steady state constant exciting force.

The results are obtained for the concentrated mass in non-dimensional form at different positions along the length of the beam. The same has been done for different concentrated mass ratios ($M/\rho A_c L$).

Results are obtained for columns without elastic foundations with concentrated mass at the centre of the column.

3.2 Finite Element Solution

Consider a beam on an elastic foundation subjected to a steady state constant exciting force. It is required to determine the response of the beam. The partial differential equation of the beam is given by Eq.2.1. The corresponding equation for the finite element analysis is given by Eq. 2.17. Wilson-θ technique is used for discretisation in time domain.

A general purpose finite element programme is developed which can handle different boundary conditions, initial conditions, types of loadings etc.

Also a programme is developed for columns which can handle different boundary conditions, initial conditions, types of loadings etc.

The following data is used for beams and elastic foundations.

Young's Modulus, E	$= 2.1 \times 10^6$ kg/cm^2
Moment of Inertia of the beam, I	$= 1819.3$ cm^4
Spring constant of the soil media, K	$= 525$ kg/cm^2
Mass Density of the Soil media, ρA_c	$= 1.62$ g/cm^3
Damping Ratio, D	$= 0.10$
Length of the beam, L	$= 1000$ cm

3.3 Verification of Finite Element Solution

The finite element solution for an infinite beam resting on an elastic foundation and carrying a concentrated mass at the centre of the beam is verified with the already existing solution

by Jain [11], for a concentrated mass ratio of $\alpha = 0.34$ and it can be seen from the Fig.3.1 that the results are in good agreement with the results available. Also the results for beams without elastic foundation are verified as can be seen from Figs. 3.2, 3.3 for simply supported and fixed beam cases respectively.

The Response versus Time curve at the midpoint of the beam resting on elastic foundation, is plotted in Fig.3.4 for a concentrated mass ratio of $\alpha = 0.34$, with a steady state exciting force of $F_o \cos \omega t$ where $F_o = 210$ kg. Similarly the Response versus Time curve for a concentrated mass ratio of $\alpha = 0.0$ is plotted in Fig.3.5. It can be clear from these two figures that there is a decrease in the response with the presence of concentrated mass. Also Figs.3.6 and 3.7 show the variation of Response versus Time when the exciting forcing function is $274 \cos \omega t$, with and without concentrated masses respectively. It has been observed from these figures also that the response decreases in the presence of the concentrated mass.

In Fig.3.8, the transient response is plotted for a beam on Elastic foundation with no damping in the system, for various points along the length of the beam. The dimensionless concentrated mass ratio α is 0.1 and the steady state constant exciting force is $274 \cos \omega t$ at resonant frequency. The mass position is varied along the length of the beam, each curve representing a particular position from the left end of the beam as denoted in the Fig.3.8. It can be seen from this figure that when the mass position is 1/10 th of the beam from the left end

the response is higher than that of the no absorber case for points close to the centre of the beam and is almost negligible for points away from the midpoint of the beam. This is true for the mass position at 2/10 ths from the left end of the beam. But the response is almost same for mass positions x = 300 cms and 400 cms from left end of the beam. For the mass position at the mid of the beam the response decreases and the rate of decrease is high as we move along points close to the centre of the beam.

The same pattern has been observed for $\alpha = 0.2$ (Fig.3.9) case also, but when the absorber is placed at x = 400 cms from the left end of the beam, there is a reduction in response, the reduction being almost equal to the reduction observed in the mass position x = 500 cms.

For $\alpha = 0.3$ (Fig.3.10) it is observed that there is reduction in response even for mass positions from x = 200 cms onwards. The decrease in response increases as we move towards the centre of the beam. Similarly, the response curves are plotted for $\alpha = 0.4$ and $\alpha = 0.5$ in Figs.3.11 and 3.12 respectively.

Fig.3.13 shows the variation of transient response at various points along the length of the beam at various positions of the concentrated mass for $\alpha = 0.10$ with a steady state constant exciting force of 274 cos ωt of resonant frequency. Here the nature of response reduction at various points is somewhat arbitrary, but when the concentrated mass is at the centre of the beam there is a reduction in the response and this reduction in

response increases as we move close to the centre of the beam. Similarly, for the concentrated mass ratios of α = 0.2, 0.3, 0.4, 0.5, responses are plotted at various points along the length of the beam, in Figs.3.14 to 3.17, respectively. From these figures also it can be seen that the response is reduced when the mass position is at the centre of the beam and this response reduction increases as we move towards points close to to the centre of the beam.

Fig.3.18 shows the variation of the transient response at the midpoint of the beam with different values of α, and with the absorber at the midpoint of the beam. It can be clear from this graph that there is a considerable amount of reduction in response as the value of α increases. For α = 0.9, the vibration is almost reduced by 50% due to the presence of this mass at the centre of the beam.

Response is also plotted for a column with one end free and one end fixed, without elastic foundation with concentrated mass ratios of α = 0.0 and α = 0.34 and it is seen from Fig.3.19 that the response is reduced at the free end of the beam.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

4.1 Conclusions

Based on the results obtained in this investigation the following conclusions can be drawn :

1. From the analytical results on the beam on elastic foundation carrying a concentrated mass on it and subjected to a steady state dynamic force, it can be seen that the effect of the concentrated mass is quite pronounced, for greater values of concentrated mass ratios.
2. The maximum displacement decreases very fast as the concentrated mass value increases on the beam.
3. The results obtained are in good agreement with the already existing results for beams carrying concentrated masses and resting on elastic foundation at the centre of the beam.
4. When the mass position is at the centre of the beam, higher decrease in displacements are observed.
5. For columns without elastic foundation also the influence of the concentrated mass is observed on the displacement function of the column.
6. The finite element method developed is very general and hence can be applied to a wide variety of beams on elastic foundation problems, just by changing input data in the programme.

4.2 Recommendations for Further Research

1. A spring mass and dashpot analysis can be done on both beams and columns on a elastic foundation case.
2. The transient response absorption is very important in earthquake kind of loading. Hence analysis for earthquake loads can be done with the absorber system used in the present work.

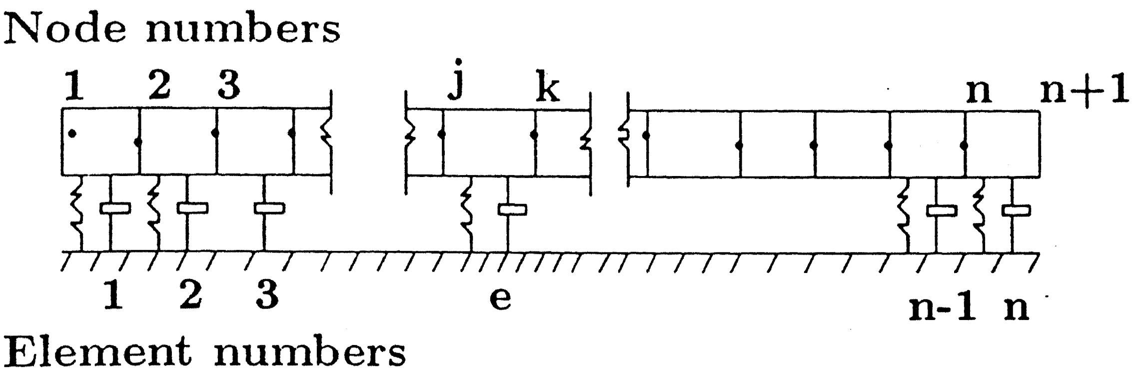

Fig. 2.1 Beam on elastic foundation divided into n finite elements.

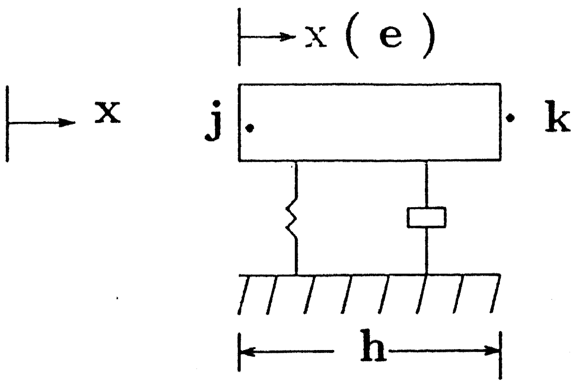

Fig. 2.2 Typical finite element of the beam on elastic foundation.

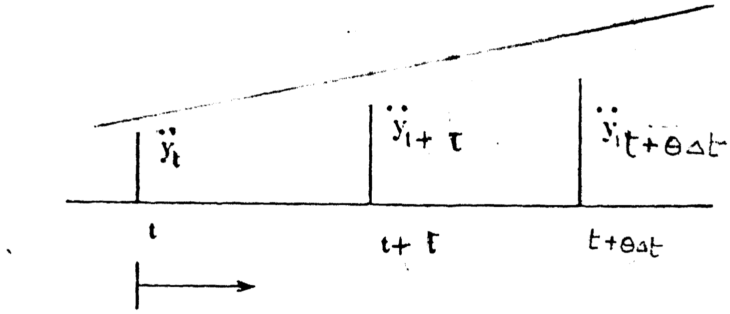

Fig. 2.3. varification of Acceliration with time (Wilson- method)

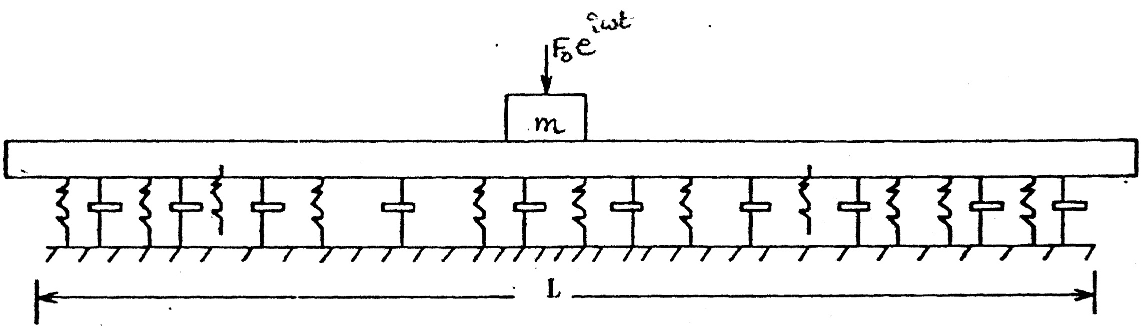

Fig. 2.4 Beam on elastic foundation subjected to a steady state force

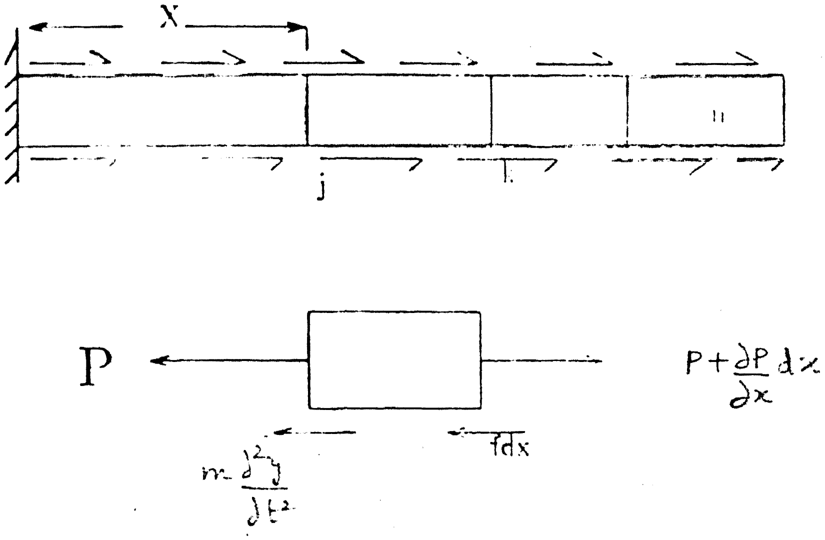

Fig. 2.5 A Column Shown with Finite element mthod

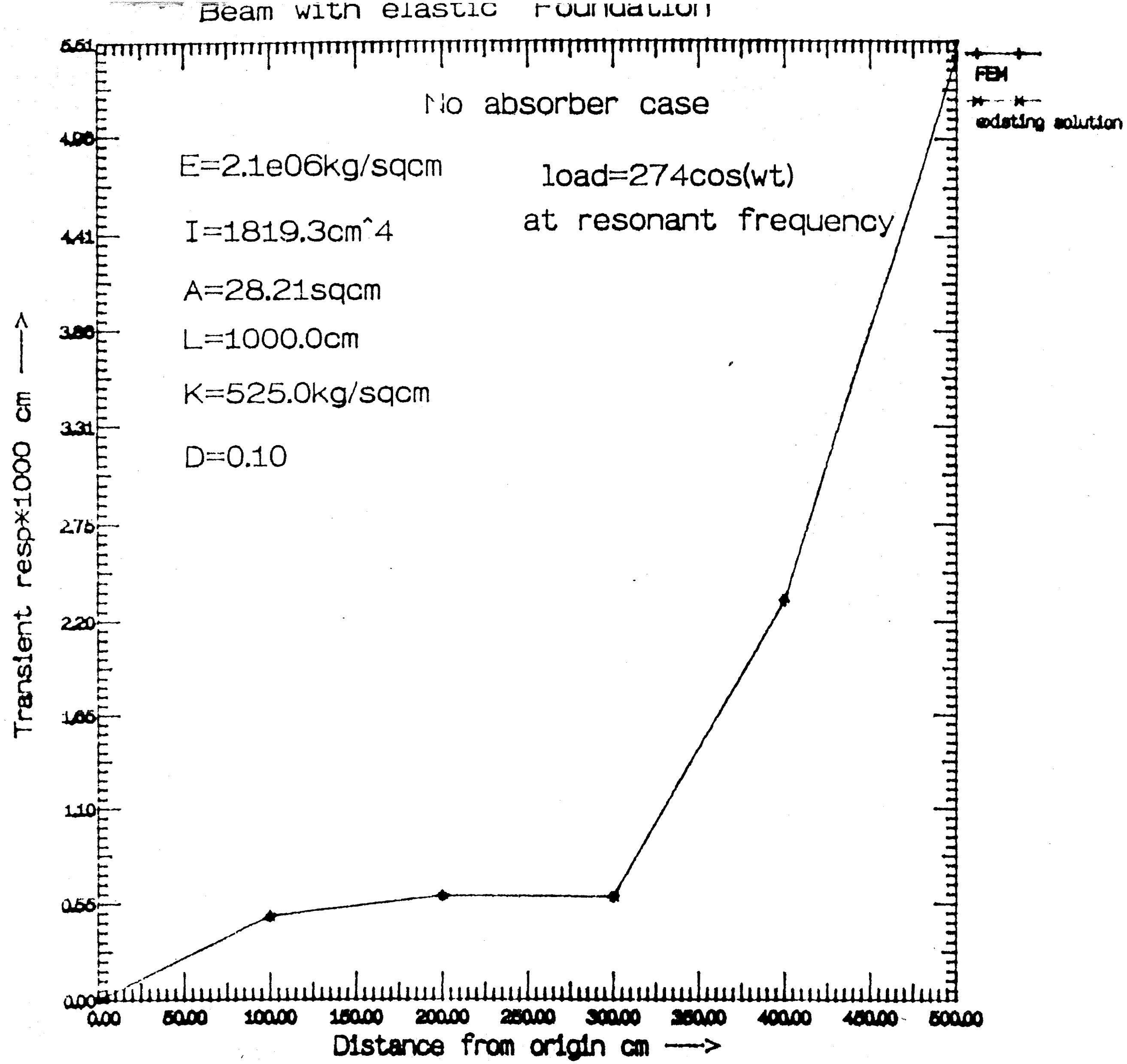

Fig. 3.1 Dispalcement along the length of the beam on elastic foundation for a steady state exciting force of 274cosωt with concentrated mass ratio of α = 0.00 (verification by existing results)

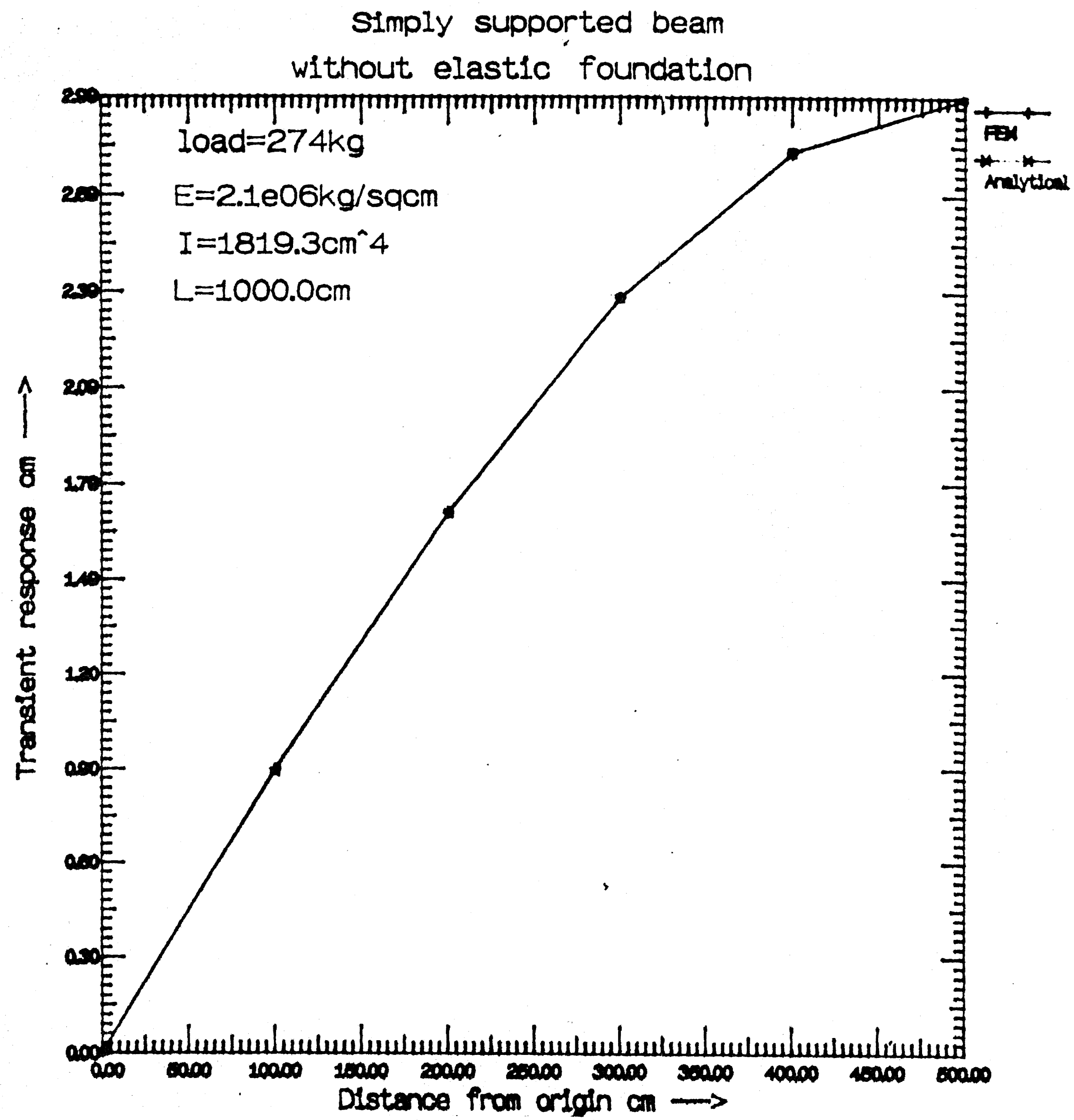

Fig. 3.2

Displacement along the length of a simply supported beam without elastic foundation.

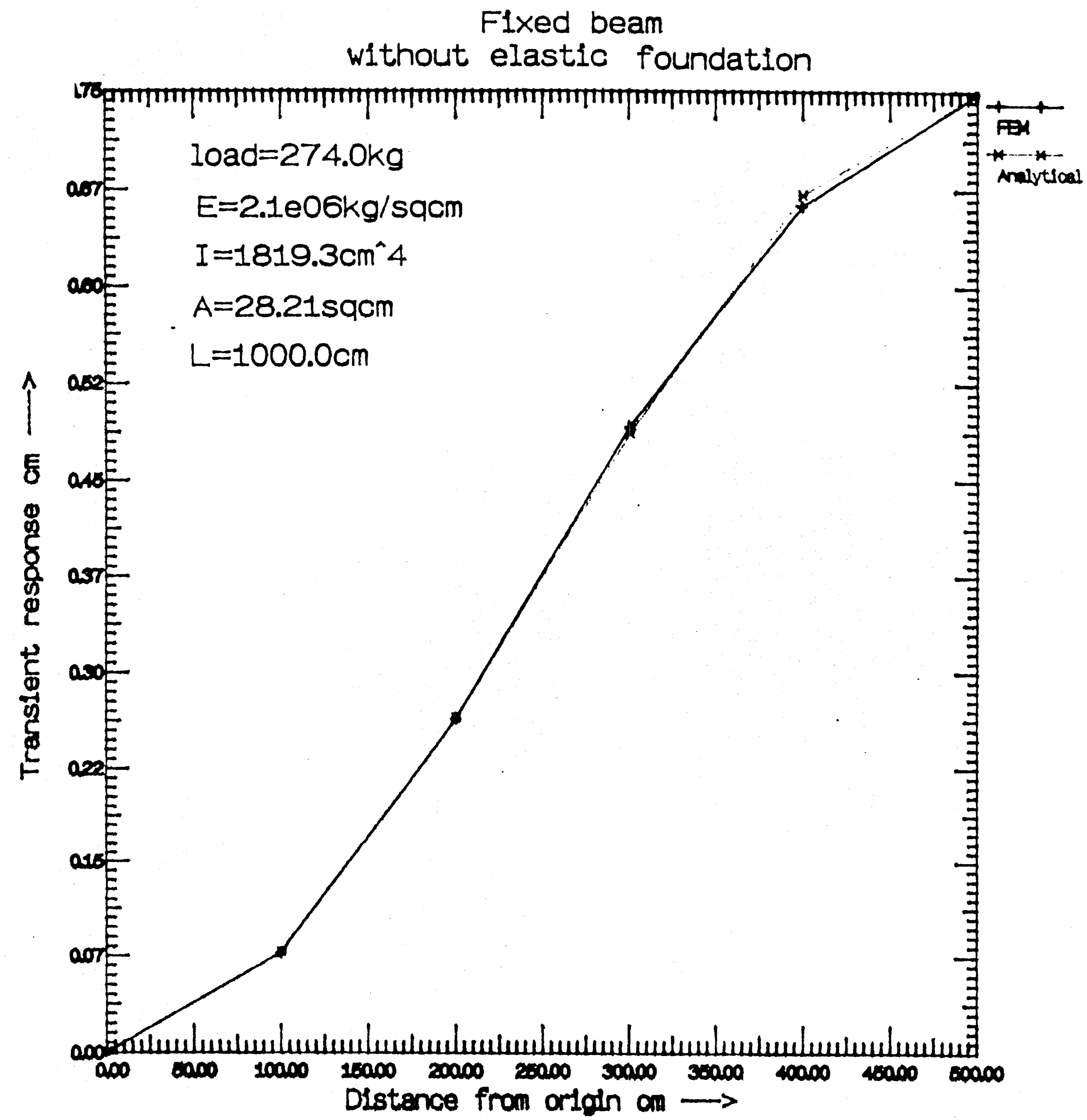

Fig. 3.3 Displacement along the length of a fixed beam without elastic foundation

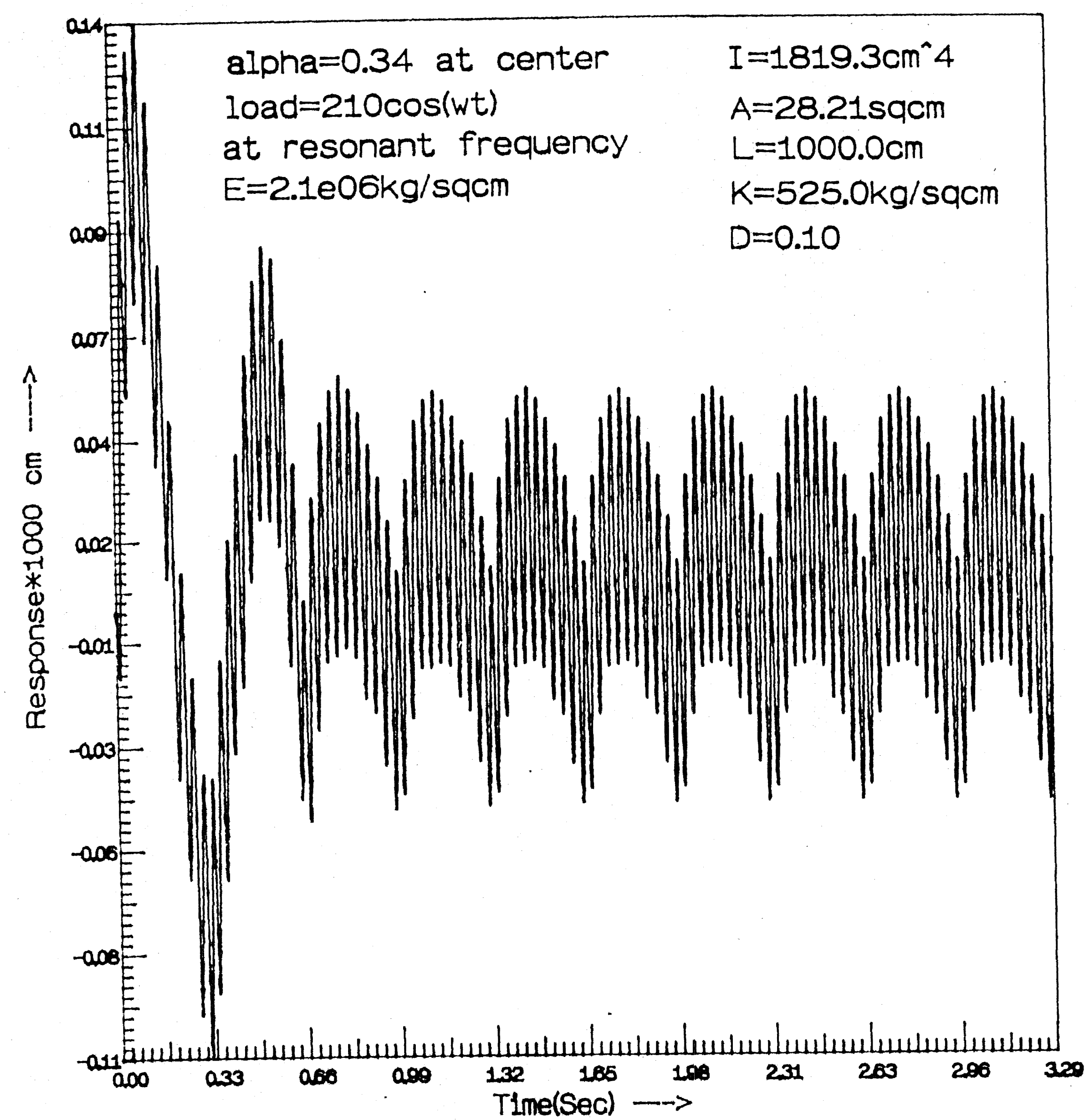

Fig. 3.4

Response vs time at the center of the beam on an elastic foundation for a steady state constant exciting force of $210\cos\omega t$ with a concentrated mass ratio of $\alpha = 0.34$.

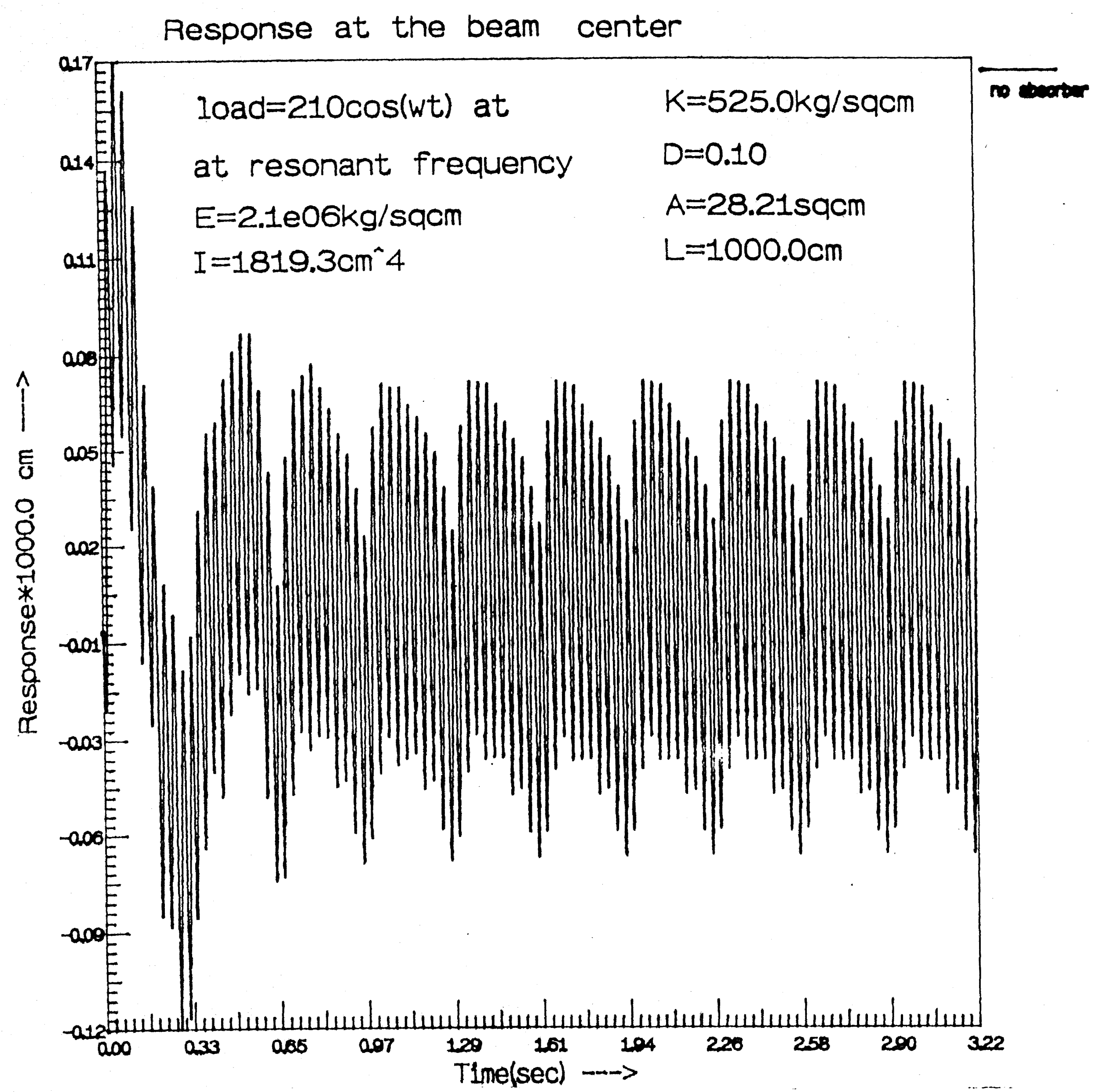

Fig. 3.5

Response vs Time at the center of the beam on an elastic foundation for a steady state constant exciting force of 210cosωt without concentrated mass.

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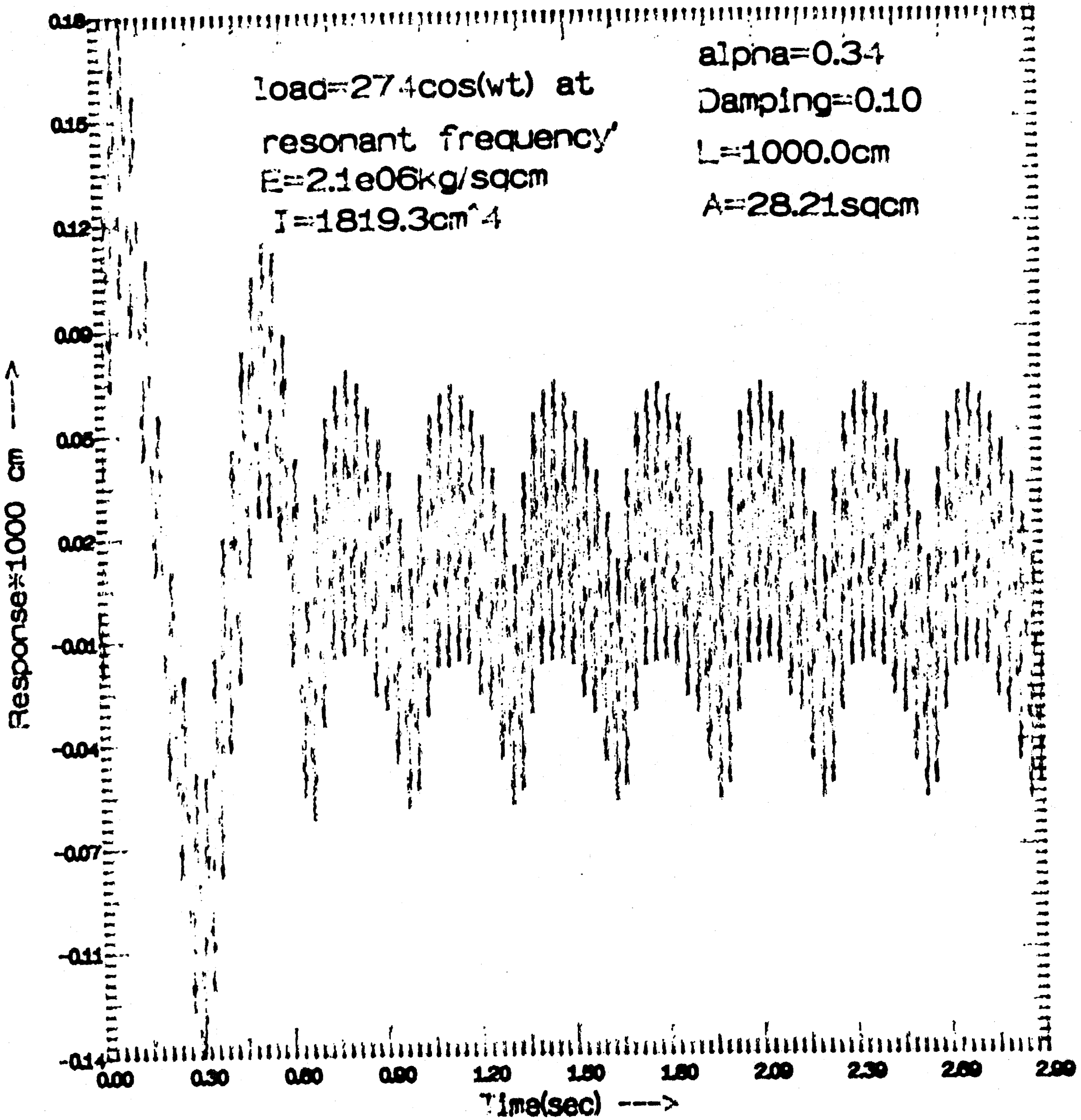

Fig. 3.6

Response vs time at the center of the beam on an elastic foundation for a steady state constant exciting force of 274cosωt with concentrated mass ratio of $\alpha = 0.34$.

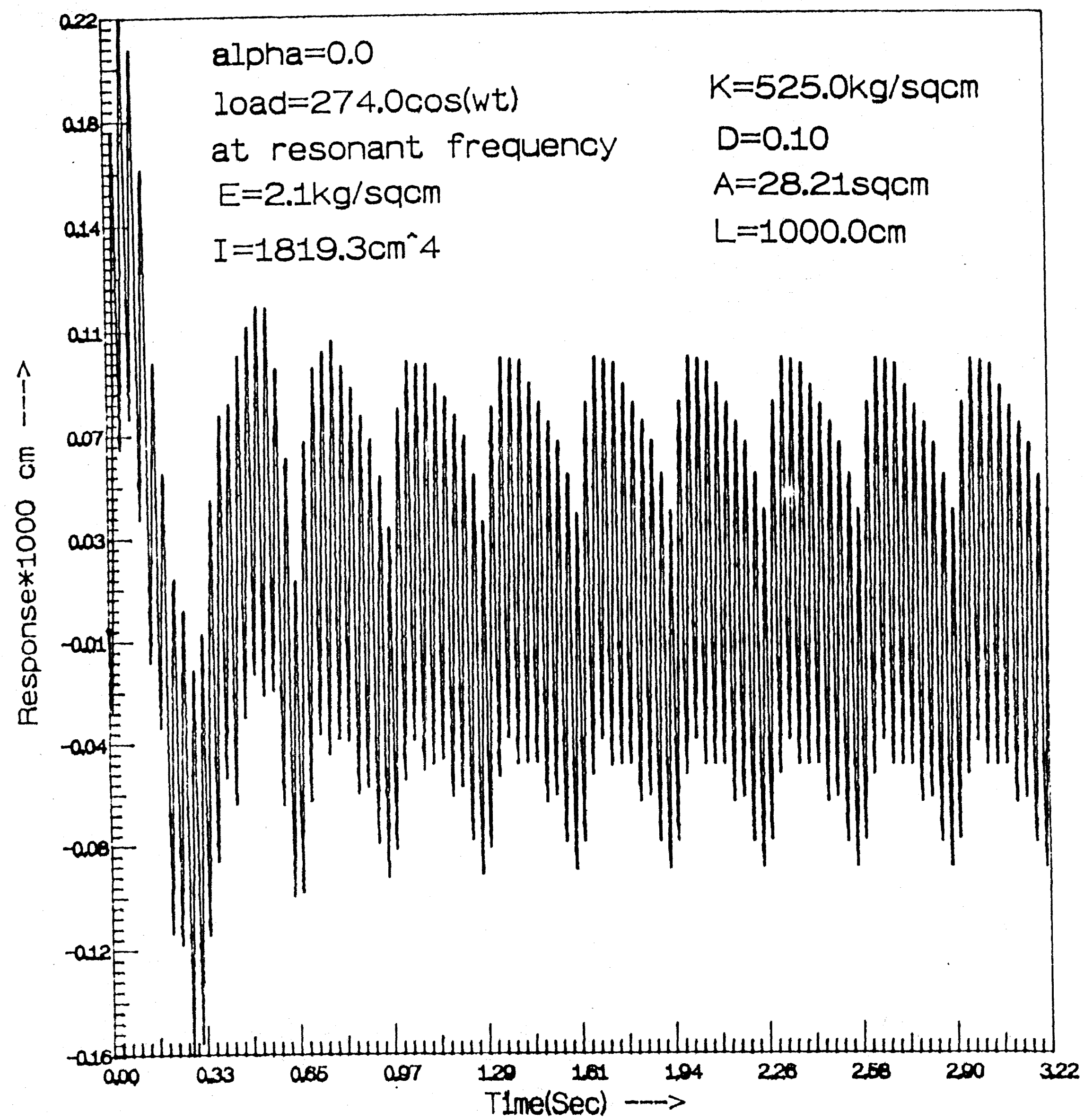

Fig. 3.7
Response vs time at the center of the beam on an elastic foundation for a steady state constant exciting force of 274cosωt without concentrated mass.

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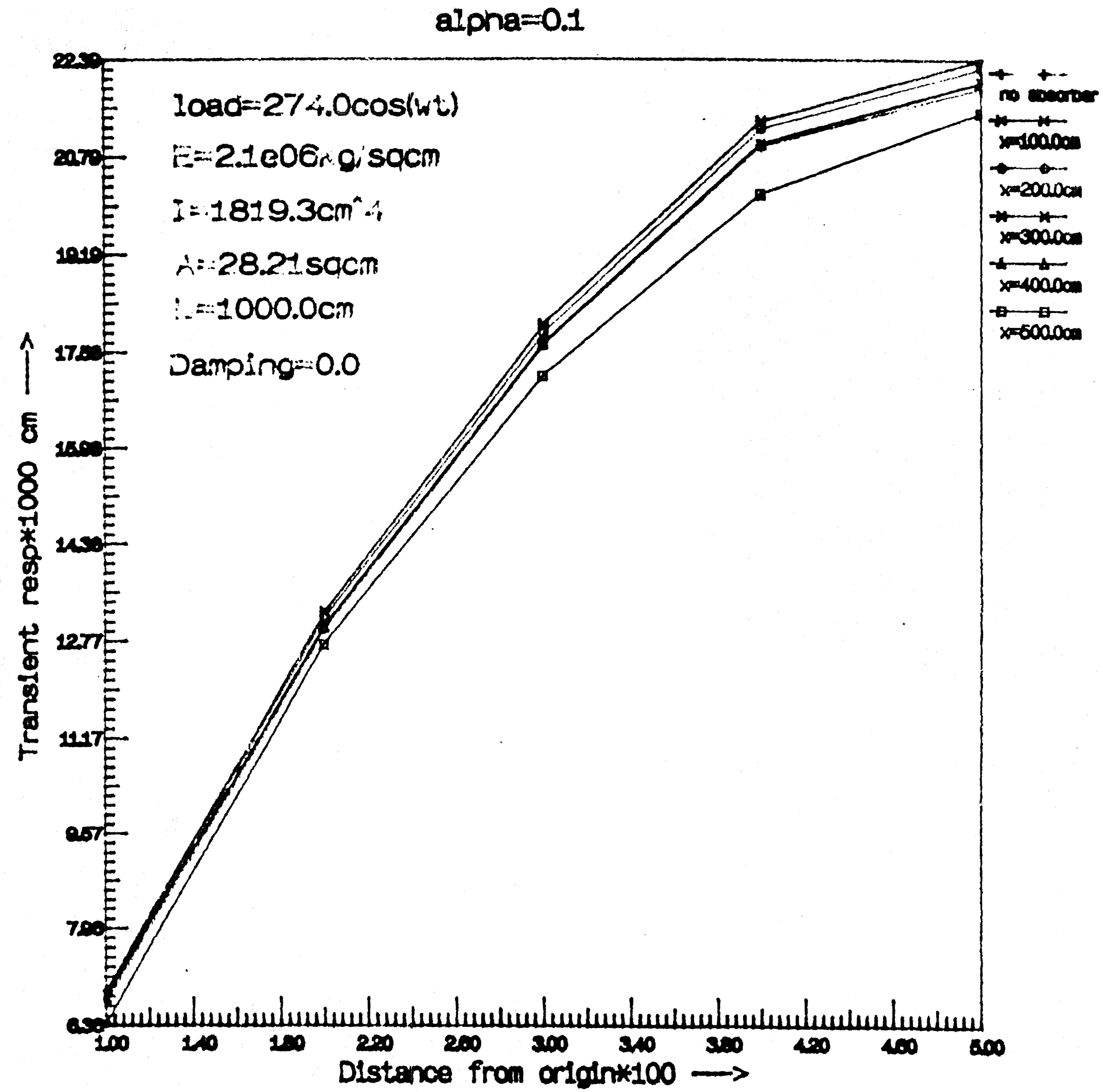

Fig. 3.8 Displacement along the length of the beam for a concentrated mass ratio of α = 0.1 for different mass positions without damping

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Plot

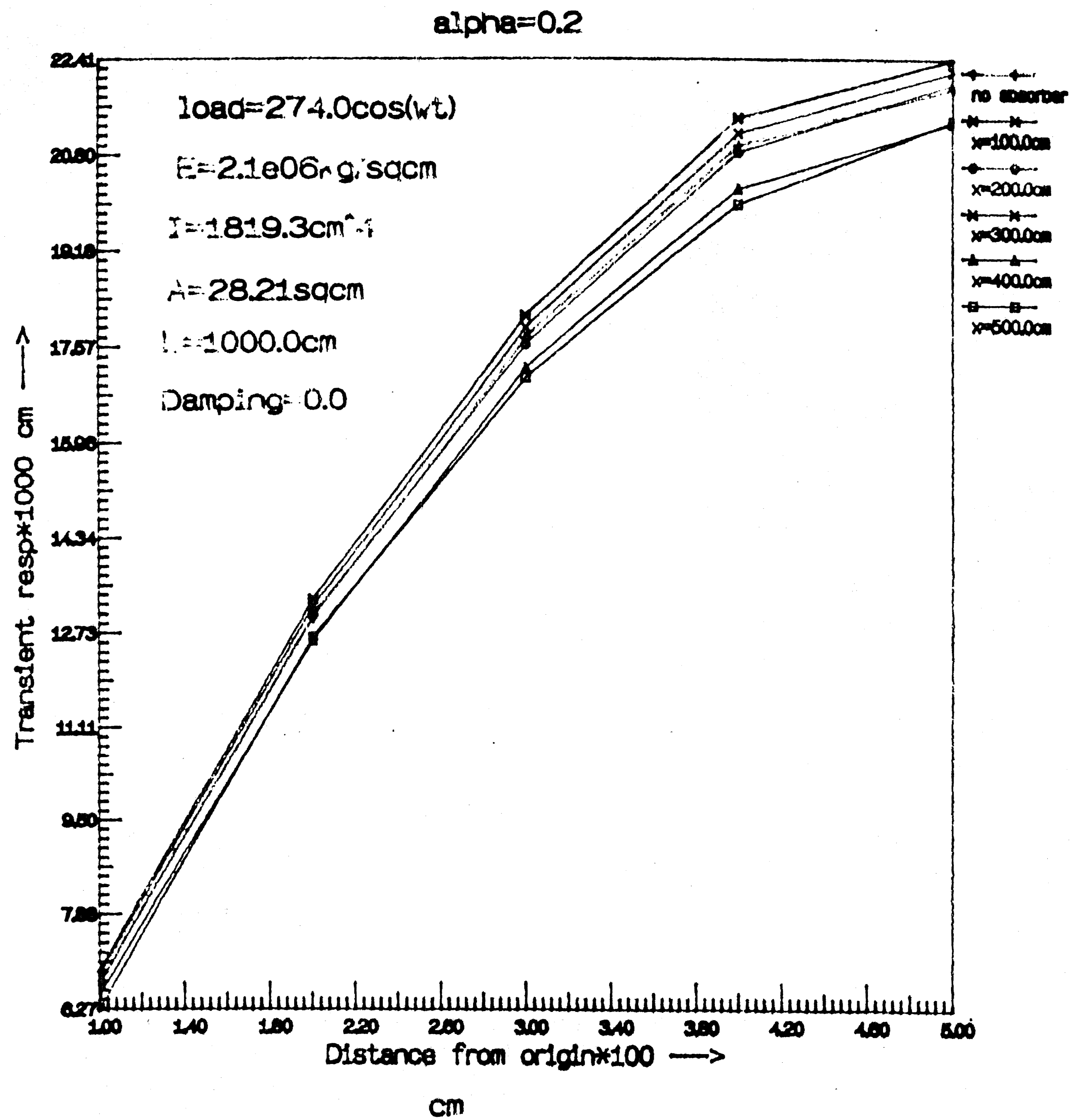


Fig. 3.9 Displacement along the length of the beam for a concentrated mass ratio of α = 0.2 for

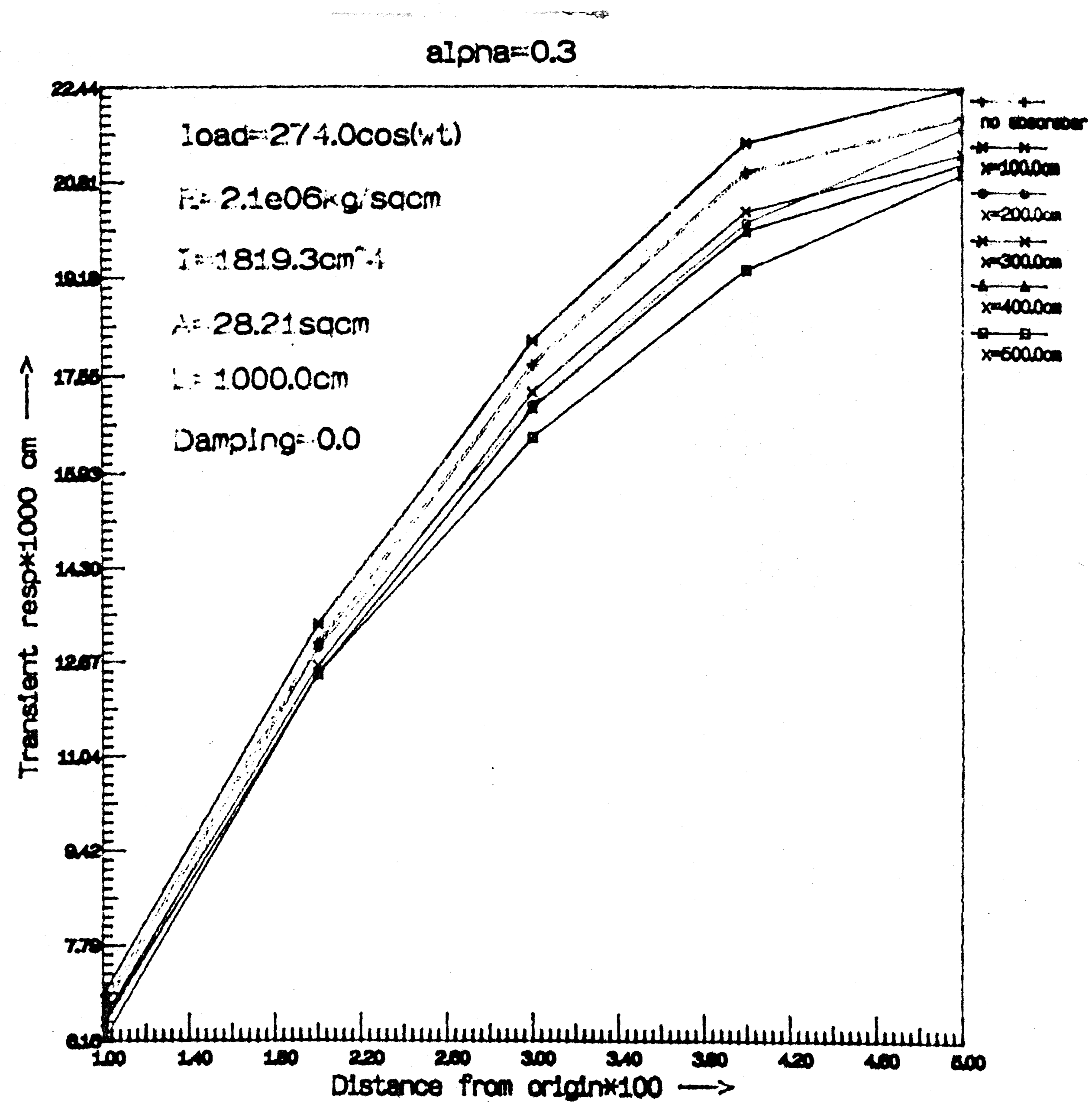
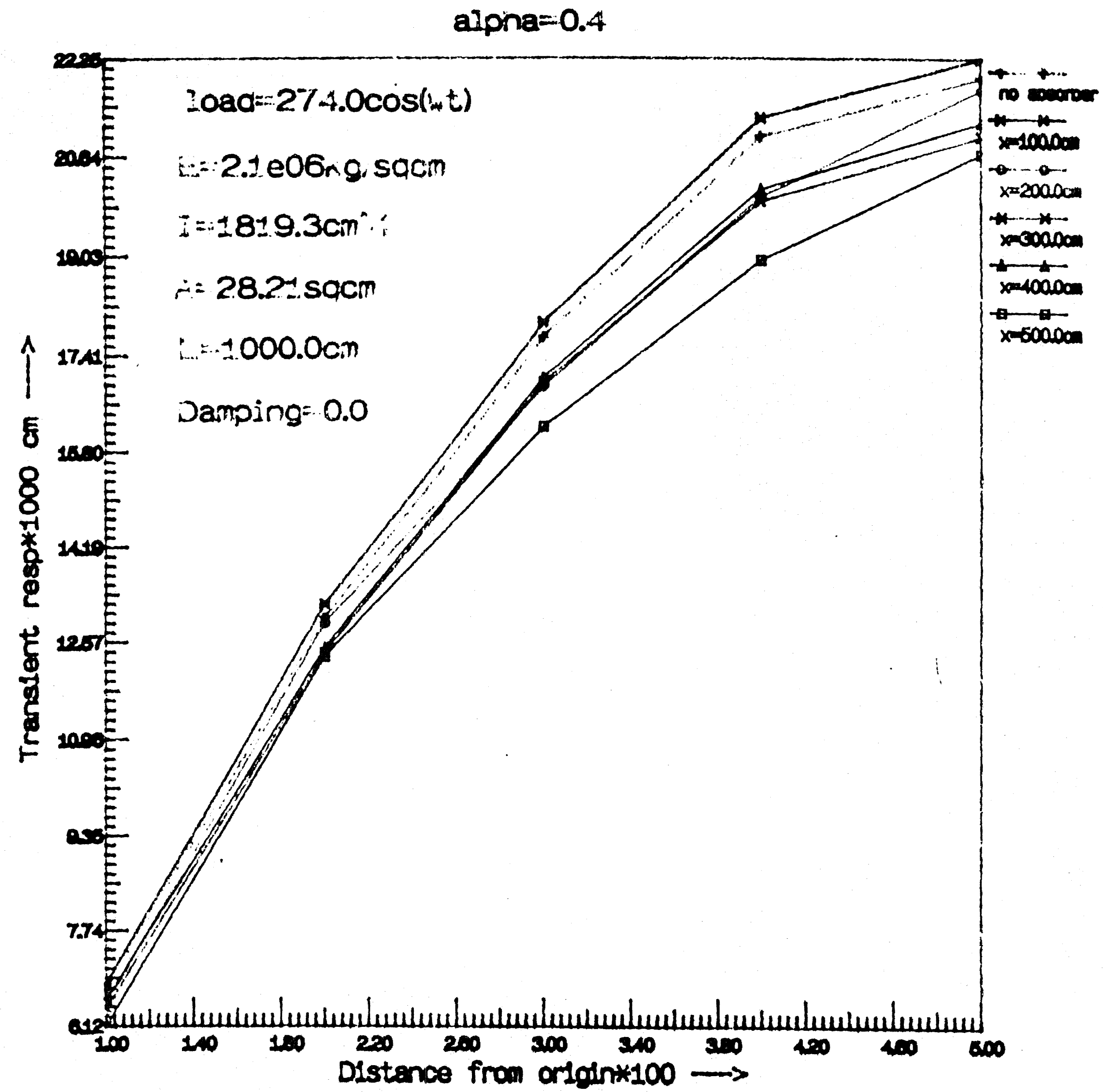


Fig 3.10 Displacement along the length of the beam for = 0.3 for



: along the length of the beam for

ted mass ratio of α = 0.4 for

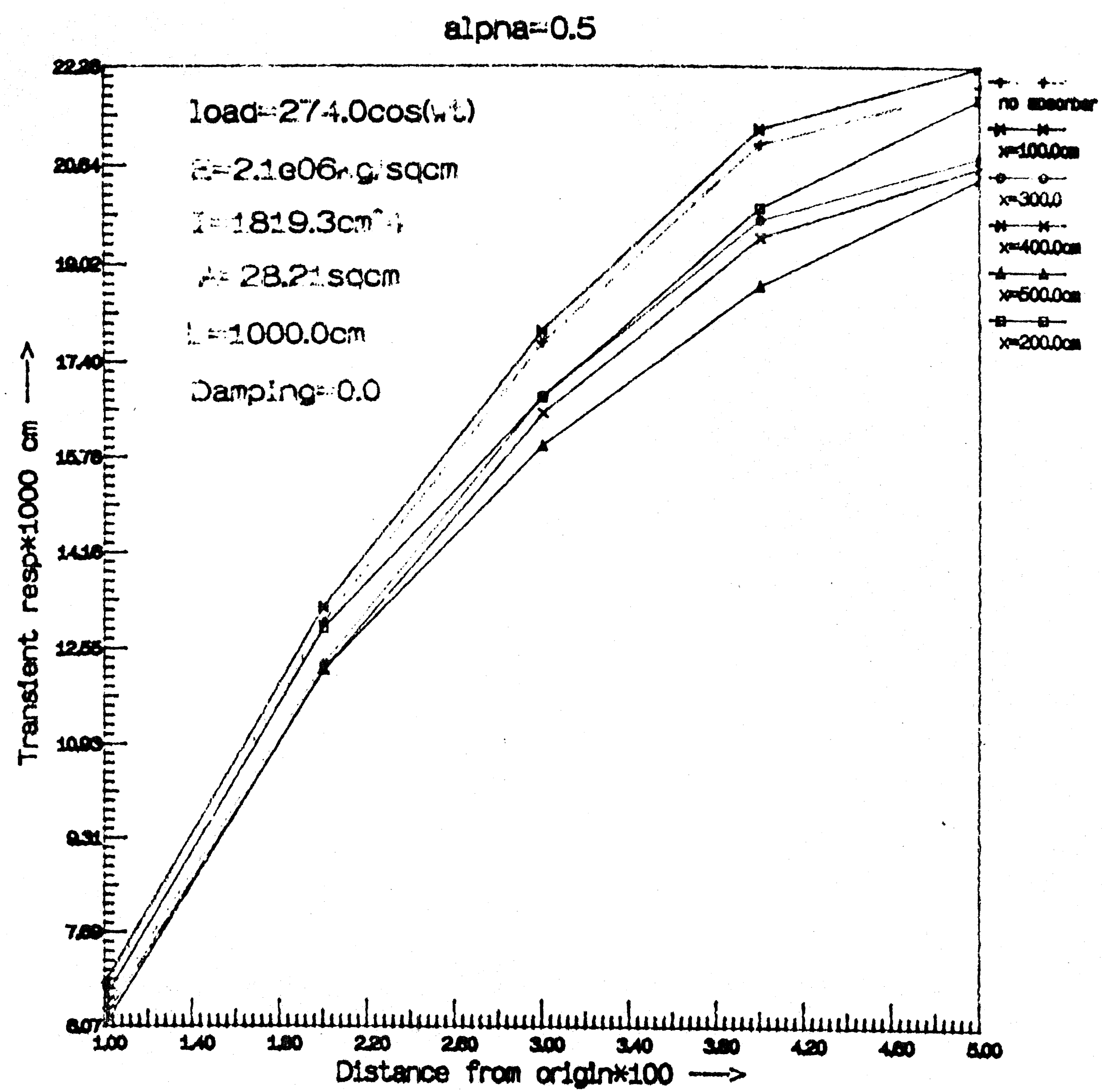


Fig. 3.12 Displacement along the length of the beam for a concentrated mass ratio of $\alpha = 0.5$ for different mass positions without damping

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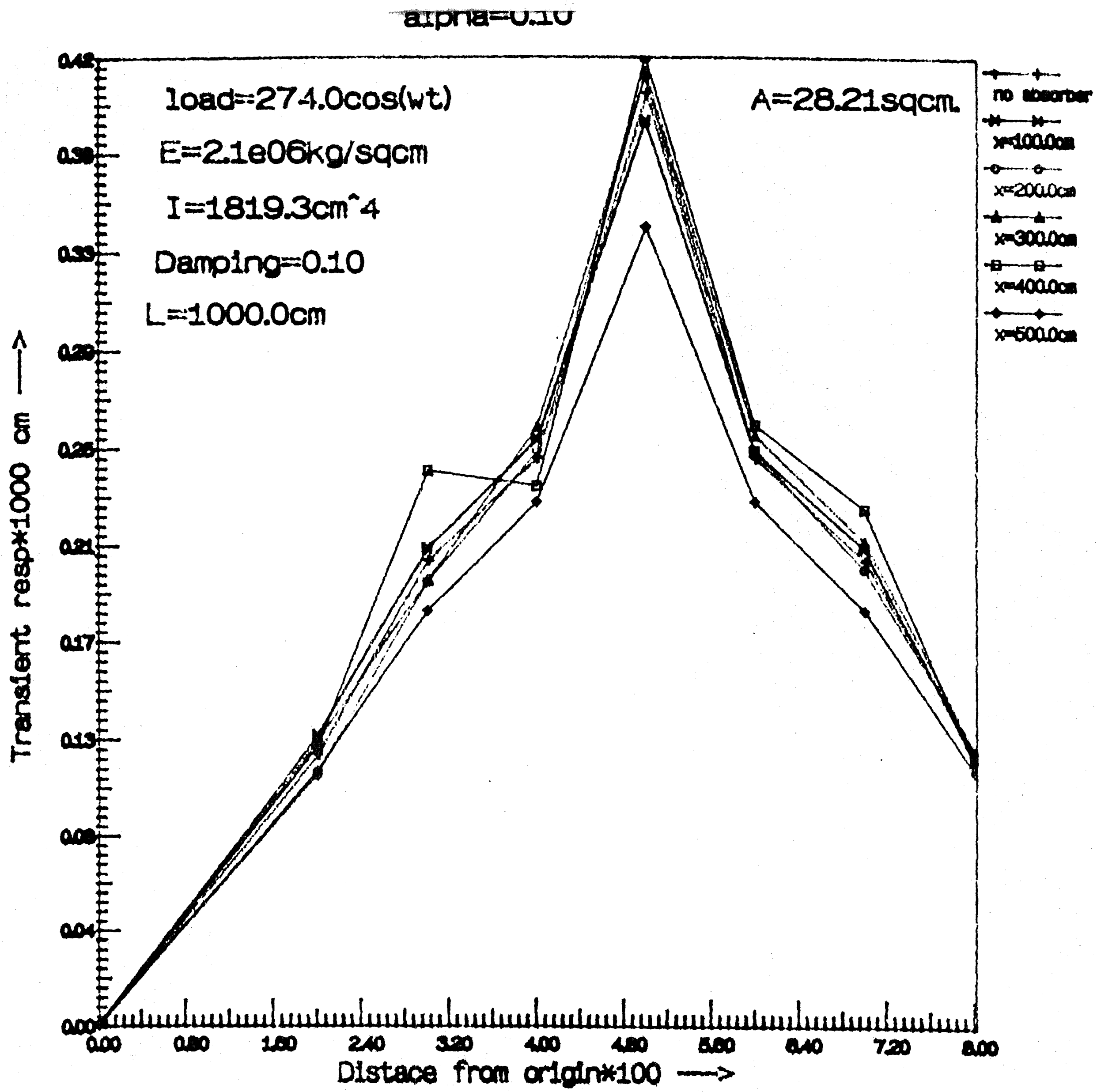

Fig. 3.13 Displacement along the length of the beam for a concentrated mass ratio of $\alpha = 0.1$ for different mass positions with damping

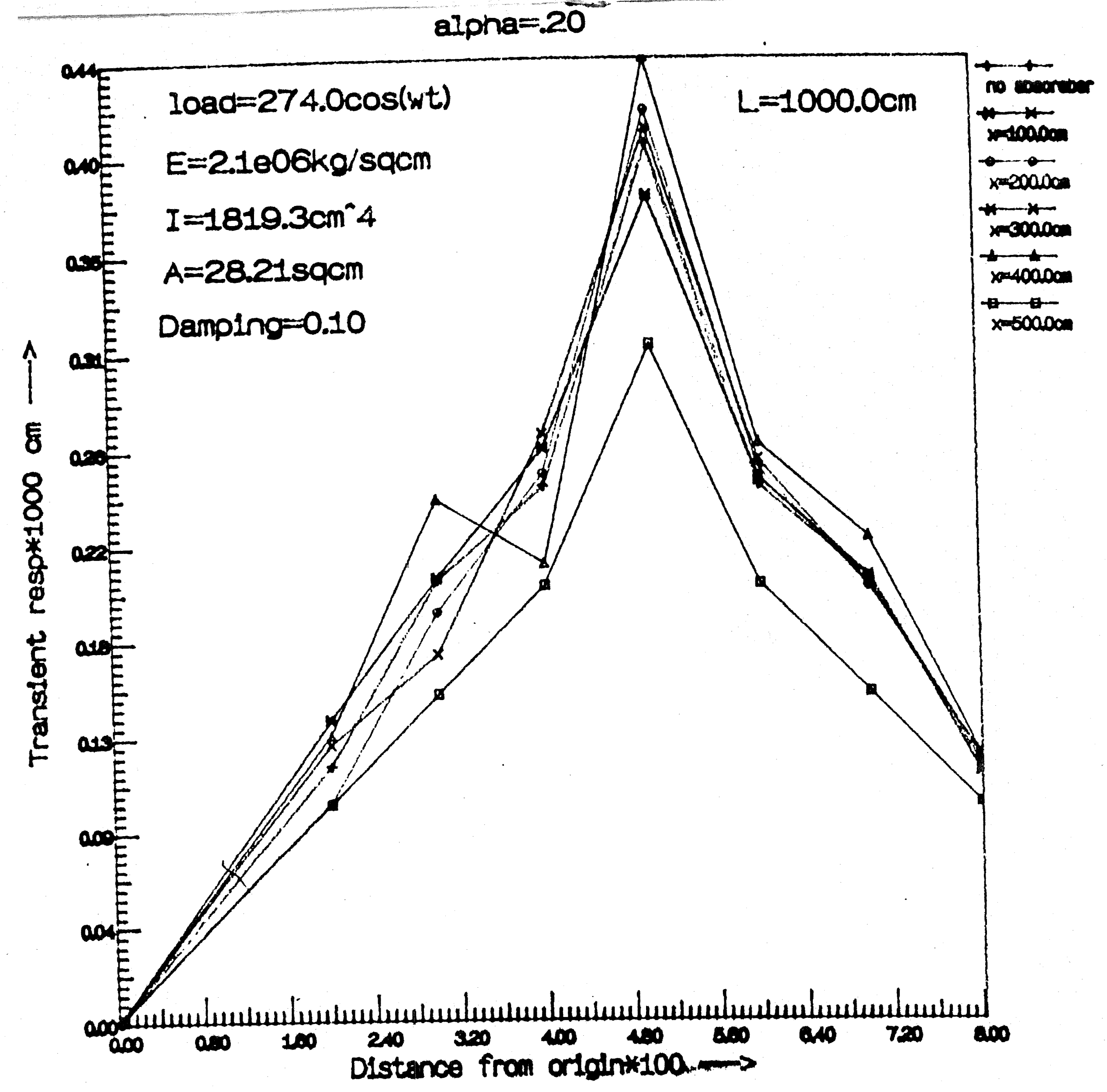

Fig. 3.14 Displacement along the length of the beam for a concentrated mass ratio of $\alpha = 0.2$ for different mass positions with damping

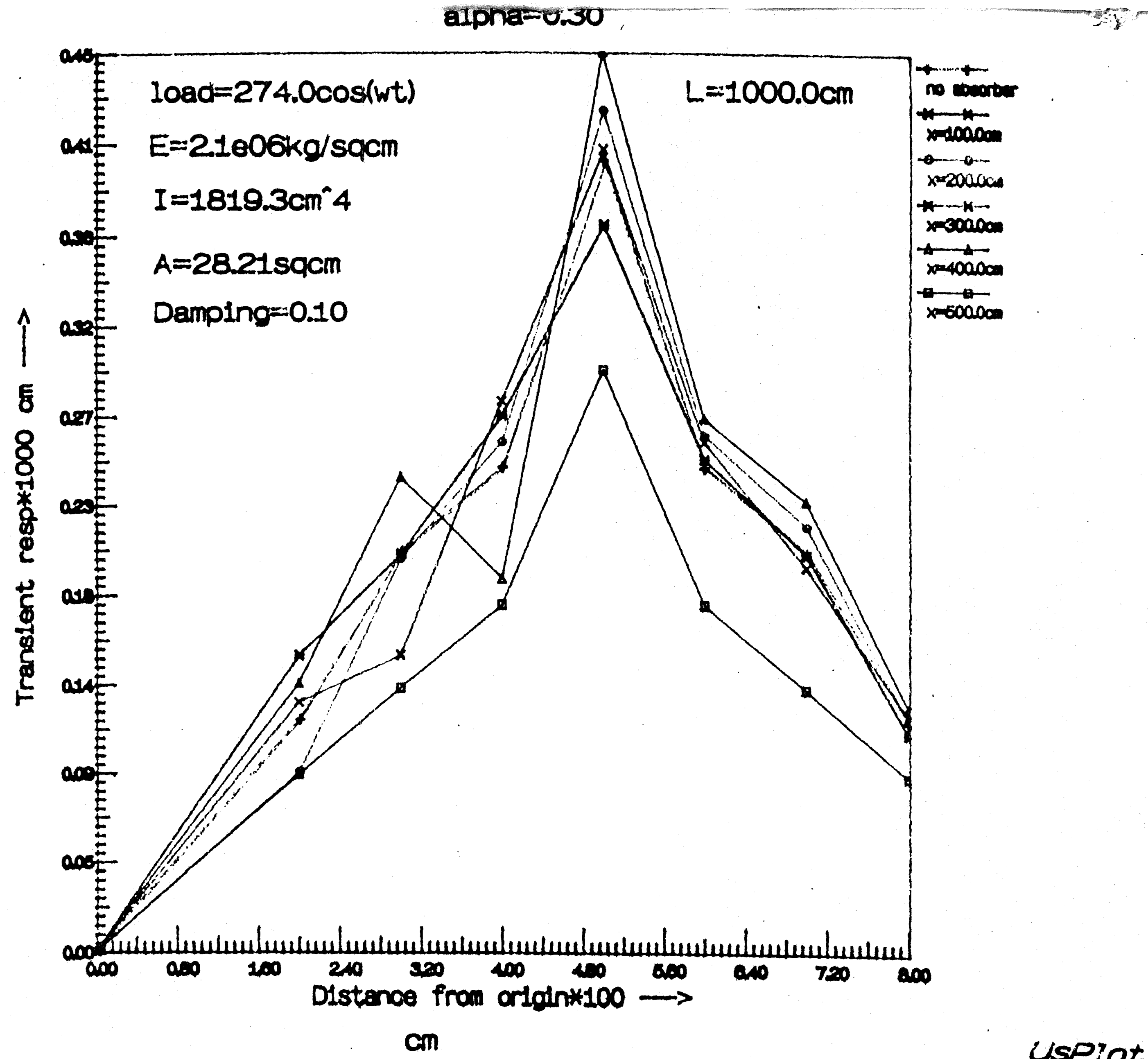

Fig. 3.15 Displacement along the length of the beam for a concentrated mass ratio of $\alpha = 0.3$ for different mass positions with damping

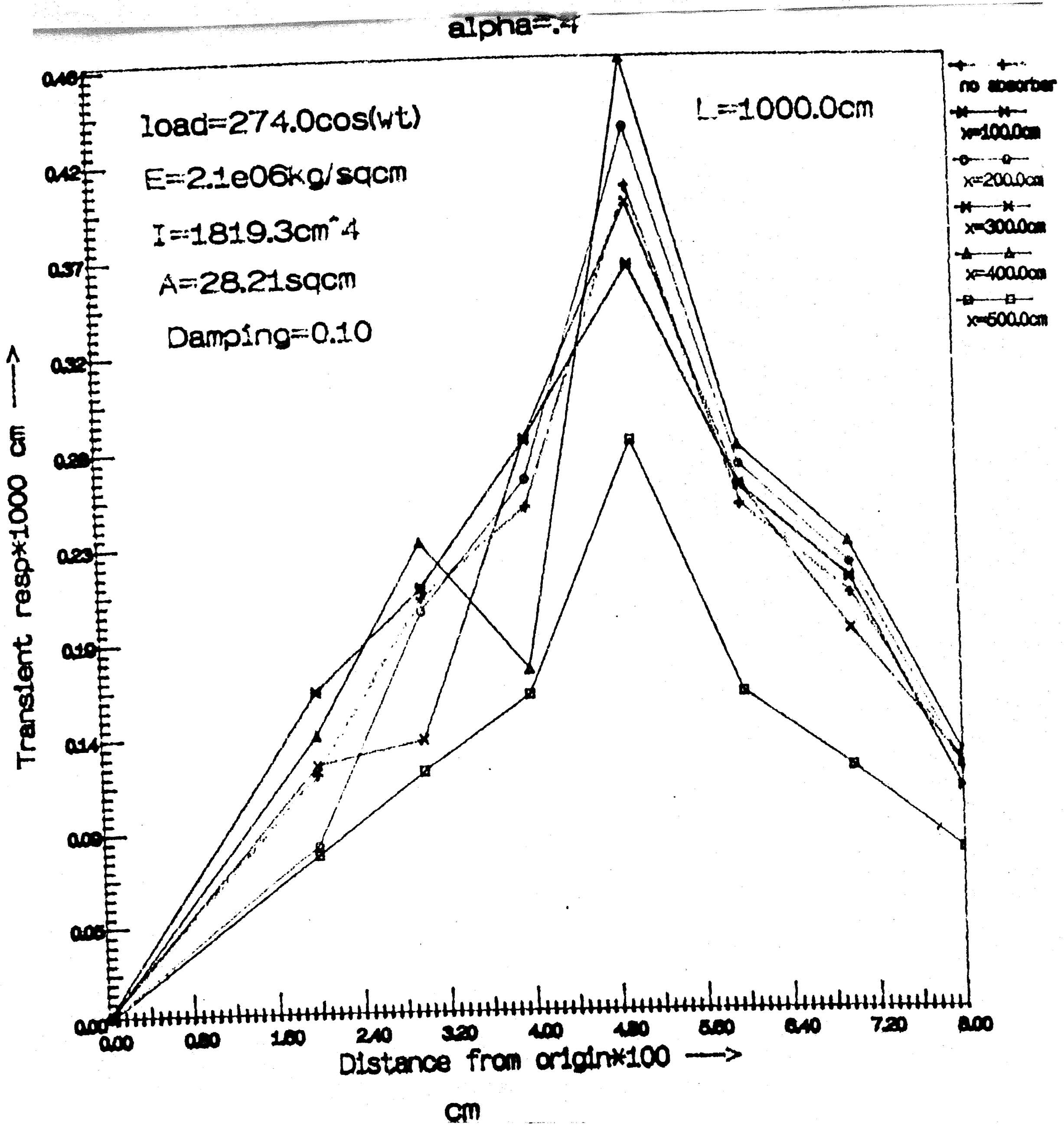

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Fig. 3.16 Displacement along the length of the beam for a concentrated mass ratio of α = 0.4 for different mass positions with damping

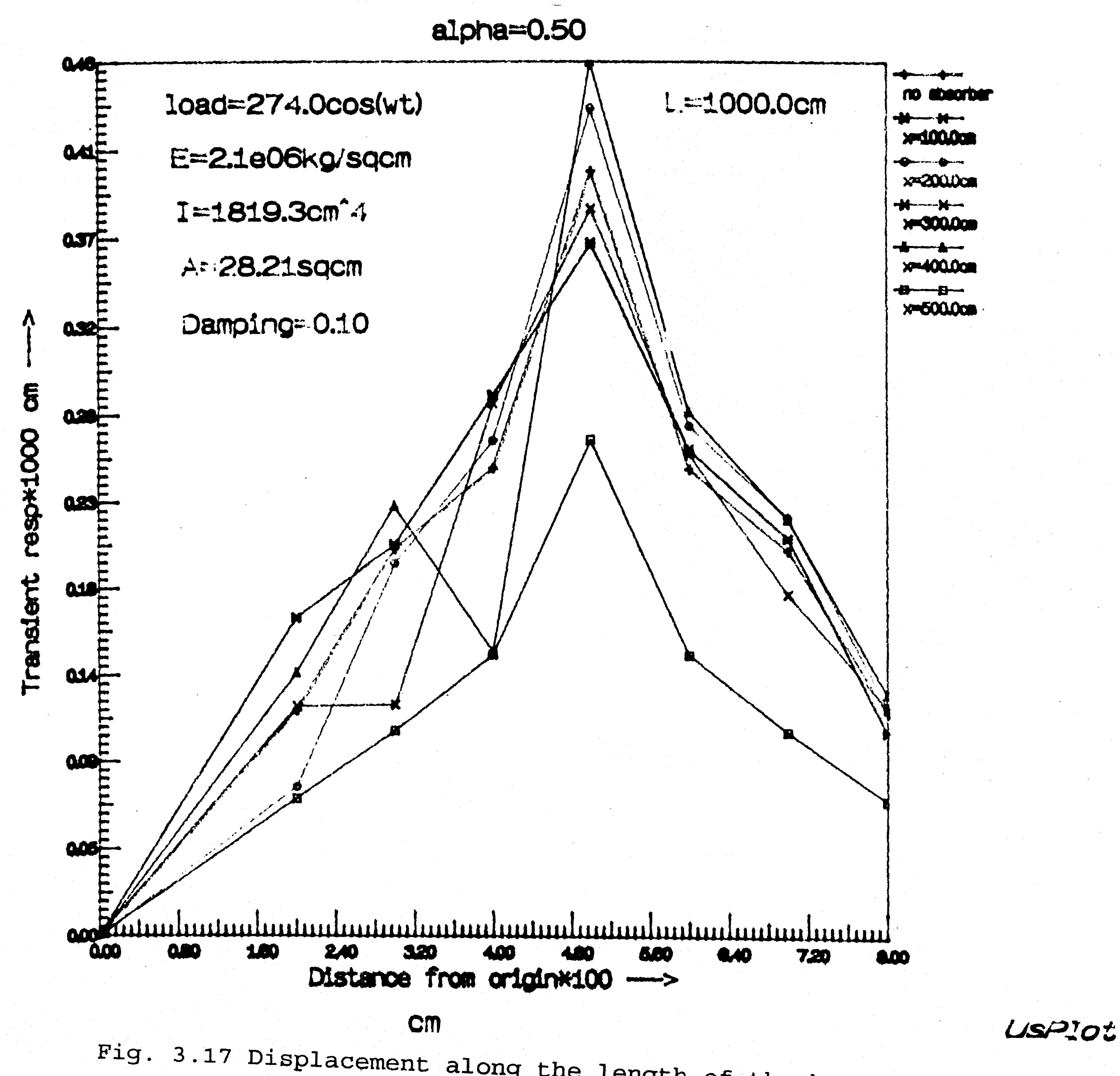

Fig. 3.17 Displacement along the length of the beam for a concentrated mass ratio of α = 0.5 for

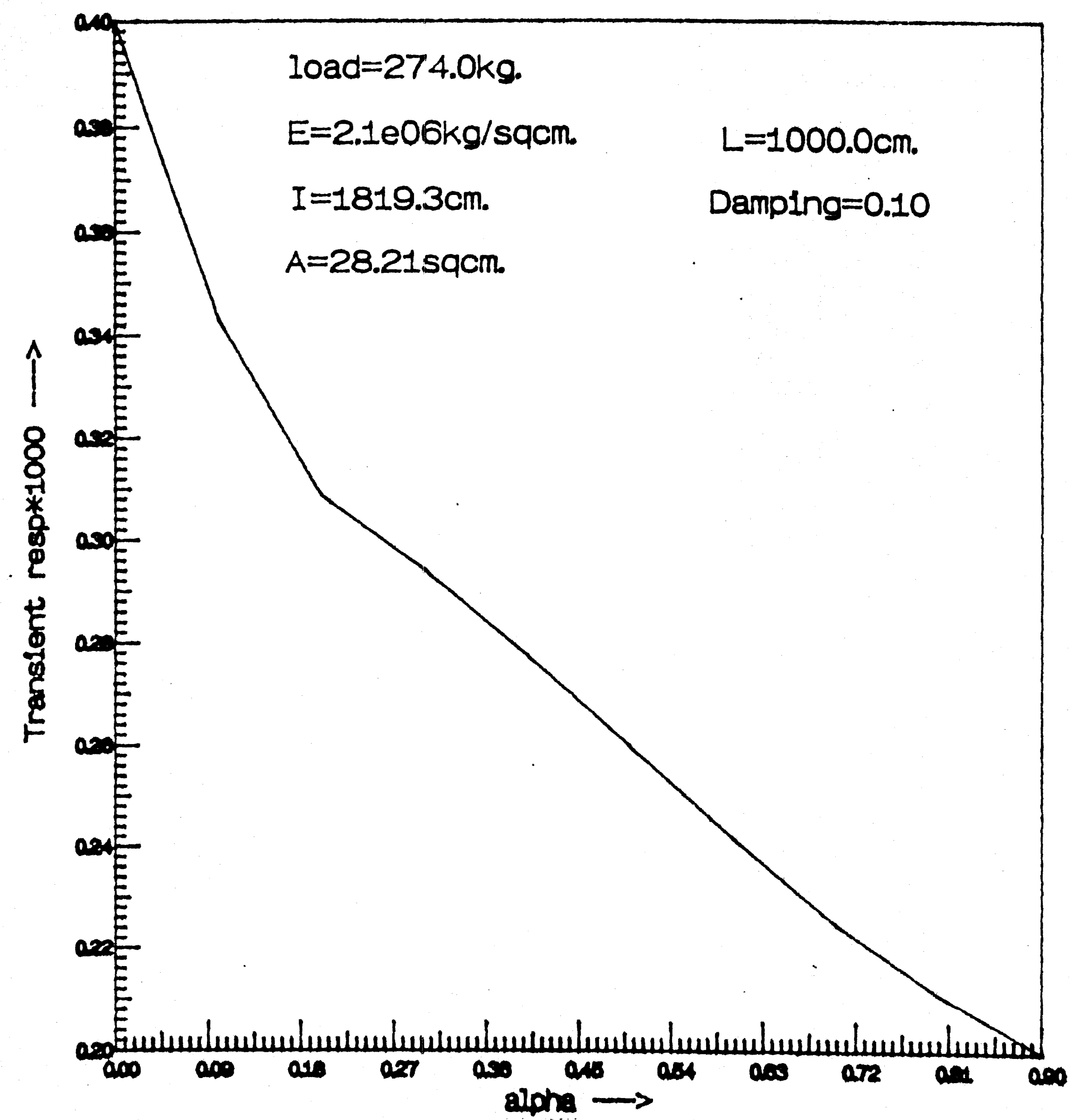

Fig. 3.18 Displacement at the center of the beam for different concentrated mass ratios with the

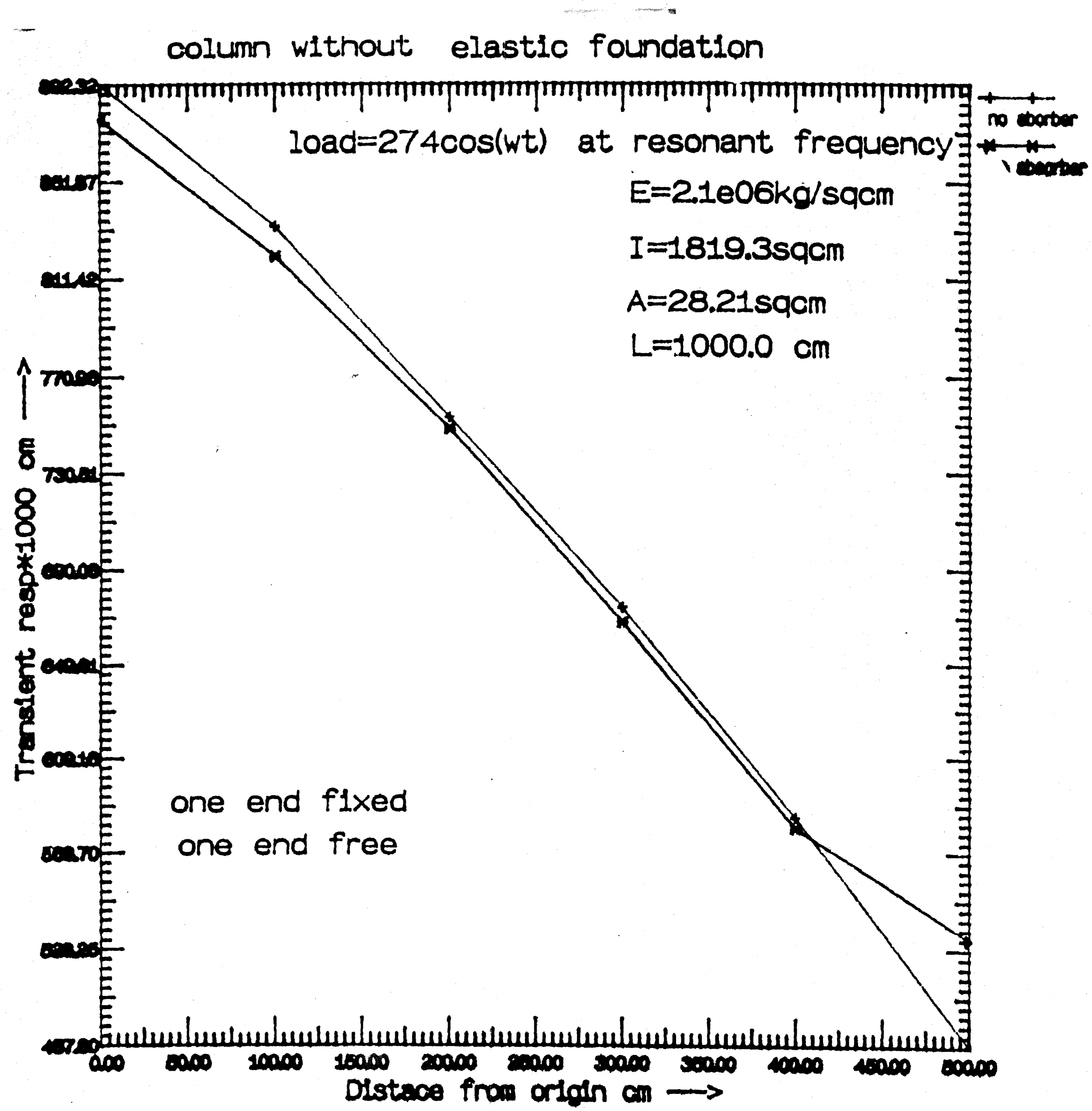

Fig. 3.19 Displacement at various points along the length of the column (one end fixed and the other end free) with and without concentrated

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REFERENCES

1. Baker, W.E. *Vibration frequencies for uniform beams with central masses*, Journal of Applied Mechanics, Vol 31, No. 2, 1964, pp.(335-337).

2. Biot, M.A. *Bending of an infinite beam on an elastic foundation*, Journal of Applied Mechanics, VOl. 4, No. 1, ASME 1937, pp. (A1-A7).

3. Boley, B.A & Chao, C.C, *Some solutions of the Timoshenko Beam Equations*. Journal of Applied Mechanics, Vol. 22, No. 4, Dec 1955, pp. (579-586).

4. Boley, B.A & Chao, C.C, *An approximate analysis of Timoshenko Beams under Dynamic Loads,* Journal of Applied Mechanics, Vol. 25, No.1, Mar 1958, pp. (31-36).

5. Chen, Y. *On the vibrations of Beams or rods carrying a concentrated mass,* Journal of Applied Mechanics, Vol. 30, Trans. of ASME, 1963, pp. (310-311).

6. Das, Y.C. & Deshmukh, R.S., *Beam Vibrations with centrally attached mass,* Bulletin of Indian Society of Earthquake Technology, Vol. VII, No. 2, June 1970, pp. (89-95).

7. Fletcher, D.Q. & Herrman, L.R., *Elastic Foundation Representation of continuum,* Journal of Engg. Mechanics. Div. ASCE, Vol. 97, No. EM1, Feb 1971, PP. (95-107).

8. Funstone, N.E.& Hall, W.J., *Footing Vibrations with Non linear Sub-grade Support,* Journal of Soil Mech. and Found. Div., ASCE, Vol. 93, SM5, 1967, pp. (191-211).

9. Heteny, M., *Beams on Elastic Foundation,* The Univ. of Michigan Press, Ann Arbor, Michigan, 1946.

10. Hopmann II, W.H., *Forced Lateral Vibration of Beam carrying a Concentrated Mass,* Jl. of App. Mech., Vol 19, Tran. of ASME, Series E, 1952, pp. (301-307).

11. Jain A K, *Experimental and Analytical Investigations on a Class of Dynamic Problems in Beam-Foundation Interaction,* Ph.D. Thesis, Dept. of Civil Engg., I.I.T.Kanpur, 1977.

12. Kameswara Rao, N.S.V., Das, Y.C. & Ananda Krishnan, M., *Dynamic Response of Beams on Generalized Elastic Foundations,* International Jl., of Solids and Structures, Vol. 11, Mar 1975, pp. (255-273).

13. Kausel, E., and Rosset, J. M., *Dynamic Stiffness of Circular Foundations,* Jl. of Engg. Mech., ASCE, Vol. 101, No. EM6, Dec. 1975, pp. (771-785).

14. Kerr, A. D., *Elastic and Visco-elastic Foundation Models,* Jl. of App. Mech., Vol. 31, No. 3, 1964, pp. (491-498).

15. Rades, M., *Steady State Response of a Finite Beam on Pasternak Type Foundation,* Int. Jl. of Solids and Structures, Vol. 6, 1970, pp. (739-756).

16. Saito Hideo & Tomizo Murakami, *Vibrations of an infinite Beam on an Elastic Foundation with Concentrated Mass of a Foundation,* Bulletin of JSME, Vol. 12, No. 50, 1969, pp. (200-205).

17. Sarrajin, M.A., Roset, J.M. & Whitman, R.V., *Dynamic Soil Structure Interaction*, Jl. of Structural Div., ASCE, Vol. 98 1972, pp. (1525-1544).

18. Srinath, L.S., and Das, Y.C, *Vibrations of Beams Carrying mass,* Jl. of App. Mech., Vol. 34, No. 3, 1967, pp. (384-385).

19. Stadler, W. & Shreeves, R.W, *The transient and Steady State Response of the Infinite Bernouli-Euler Beam with Damping and on Elastic Foundation,* Quart. Jl. of Mech. and App. math., Vol 23, No. 2, 1970, pp. (197-208).

20. Szuladzinski, G., *Discrete Model of Beams on Elastic Foundation,* Jl. of Engg. Mech. Div., ASCE, Vol. 101, No. EM6, Dec 1975, pp. (839-853).

21. Vesic, A.B, *Beams of Elastic Sub-grade and Winkler's Hypothesis,* Proc. of Fifth International Conference on Soil Mech and Found. Engg., Vol. 1, 1961, pp. (845).

22. Whitman, R. V., *Analysis of Foundation Vibrations,* Professional Paper P65-02, Soil Publications 169, Dept. of Civil Engg., M. I. T., April 1964.